Theorem: Bounding the Error in $P_n(x)$
Suppose $f$ and all its derivatives are continuous. If $P_n(x)$ is the $n^{th}$ Taylor approximation to $f(x)$ about 0, then the error for the approximation can be bounded as follows

$$|E_n(x)| = |f(x) - P_n(x)| \leq \frac{M}{(n+1)!}|x|^{n+1},$$

where $M = \max(|f^{(n+1)}|)$ on the interval between $a$ and $x$.

So, let us consider $f(x) = \arctan(x)$. How accurate is the 5\textsuperscript{th} or 10\textsuperscript{th} or 100\textsuperscript{th} Taylor approximation centered at 0 on the interval $[-1, 1]$?

Examining the graph of the $(n+1)^{st}$ derivatives, on the interval $[-1, 1]$, we can get an estimate for $M$ and so estimate the error of the $n^{th}$ Taylor approximation at $x = 1$, $E_n(1)$:

$$\frac{d^6}{dx^6} \arctan(x)$$
$M \approx 125$

So the error bound for the 5\textsuperscript{th} Taylor approximation is:

$$|E_5(1)| \leq \frac{125}{6!} = \frac{25}{144} \approx 0.174$$

$$\frac{d^{11}}{dx^{11}} \arctan(x)$$
$M \approx 3.6 \times 10^6$

So the error bound for the 10\textsuperscript{th} Taylor approximation is:

$$|E_{10}(1)| \leq \frac{3.6 \times 10^6}{11!} \approx 0.09$$

$$\frac{d^{100}}{dx^{100}} \arctan(x)$$
$M \approx 9.6 \times 10^{157}$

So the error bound for the 100\textsuperscript{th} Taylor approximation is:

$$|E_{100}(1)| \leq \frac{9.6 \times 10^{157}}{101!} \approx 0.01$$
So we see that to be within one tenth of the actual value of arctan we need a Taylor approximation of degree at least 10, and to be within one one hundredth of the actual value of arctan(1) we need a Taylor approximation of degree at least 100.

Your lab is to use the theorem above to figure out what degree Taylor approximation, centered at $a = 0$, is needed to approximate the following functions at the following values of $x$ to within one thousandth of the actual value.

<table>
<thead>
<tr>
<th>$f(x)$</th>
<th>$x$</th>
<th>Minimum Necessary $n$</th>
<th>Max Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>arctan($x$)</td>
<td>1</td>
<td>$10 \leq ? \leq 100$</td>
<td>0.0314</td>
</tr>
<tr>
<td>$e^x$</td>
<td>5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sin(x)$</td>
<td>$\pi/2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\ln(x+1)/(x+1)^2$</td>
<td>0.5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x^2 \sin(x)$</td>
<td>$3\pi/2$</td>
<td></td>
<td></td>
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</tbody>
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