Math 182: Calculus II
Error Estimates for Taylor Polynomials

Theorem: Bounding the Error in $\mathbf{P_n(x)}$

Suppose f and all its derivatives are continuous. If $P_n(x)$ is the n^{th} Taylor approximation to $f(x)$ about 0, then the error for the approximation can be bounded as follows

$$|E_n(x)| = |f(x) - P_n(x)| \leq \frac{M}{(n+1)!}|x|^{n+1},$$

where $M = max(|f^{(n+1)}|)$ on the interval between a and x.

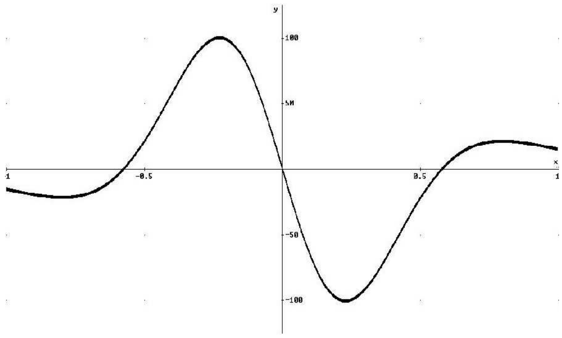
So, let us consider $f(x) = \arctan(x)$. How accurate is the 5^{th} or 10^{th} or 100^{th} Taylor approximation centered at 0 on the interval $[-1, 1]$?
Examining the graph of the $(n+1)^{st}$ derivatives, on the interval $[-1, 1]$, we can get an estimate for M and so estimate the error of the n^{th} Taylor approximation at $x = 1$, $E_n(1)$:

$\frac{d^6}{dx^6}\arctan(x)$

$M \approx 125$

So the error bound for the 5^{th} Taylor approximation is:

$|E_5(1)| \leq \frac{125}{6!} = \dfrac{25}{144} \approx 0.174$

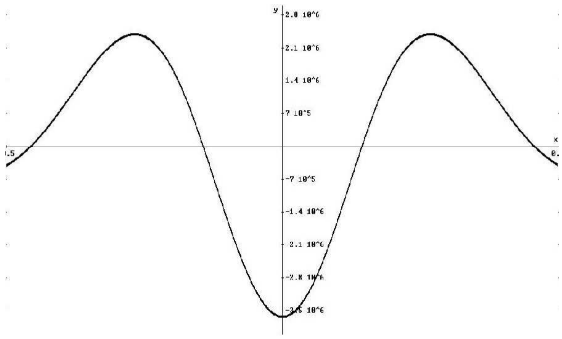


$\frac{d^{11}}{dx^{11}}\arctan(x)$

$M \approx 3.6\ 10^6$

So the error bound for the 10^{th} Taylor approximation is:

$|E_{10}(1)| \leq \dfrac{3.6\ 10^6}{11!} \approx 0.09$

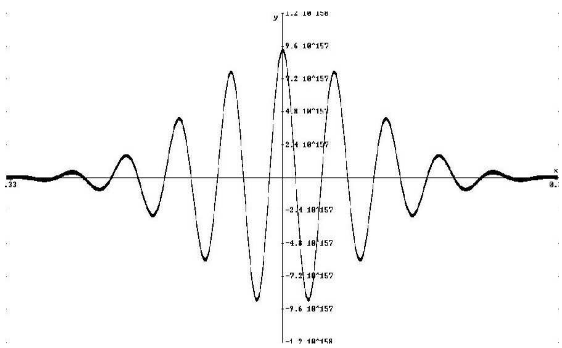


$\frac{d^{101}}{dx^{101}}\arctan(x)$

$M \approx 9.6\ 10^{157}$

So the error bound for the 100^{th} Taylor approximation is:

$|E_{100}(1)| \leq \dfrac{9.6\ 10^{157}}{101!} \approx 0.01$



So we see that to be within one tenth of the actual value of arctan we need a Taylor approximation of degree at least 10, and to be within one one hundredth of the actual value of arctan(1) we need a Taylor approximation of degree at least 100.

Your lab is to use the theorem above to figure out what degree Taylor approximation, centered at $a = 0$, is needed to approximate the following functions at the following values of x to within one thousandth of the actual value.

$f(x)$	x	Minimum Necessary n	Max Error
$\arctan(x)$	1	$10 \leq\ ?\ \leq 100$	0.0314
e^x	5		
$\sin(x)$	$\pi/2$		
$\ln(x+1)/(x+1)^2$	0.5		
$x^2 \sin(x)$	$3\pi/2$		