Introduction to the Concept of the Limit (Part I).

Our goal in this lab is to understand the idea of a limit using sequences and functions.

Consider the following *sequence* of values

$$\left\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots, \frac{1}{2^n}, \dots\right\}$$

where each term is equal to one divided by two to a power. As the the power increases the terms get smaller and smaller (since we have one divided by higher and higher powers of two), we say therefore that the *limit* as n goes to infinity of $\frac{1}{2^n}$ is 0:

$$\lim_{n\to\infty} \frac{1}{2^n} = 0.$$

In this lab we will try to decide what it really means to have a limit. Let's begin with our sequence above. Enter the sequence as the function

$$a(n) = \frac{1}{2^n}$$

Now using trial and error find a value for n so that $a(n)$ is less than $d = \frac{1}{10}$. Next find values of n so that $a(n)$ is less than $d = \frac{1}{100}, \frac{1}{1{,}000}$, and $\frac{1}{10{,}000}$. (Make a copy of this chart in your lab report.)

Question Q1. *Do you think you can always do this? That is given any little positive number d can you always find a value n so that a(n) is less than d?*

If yes, then try to describe why and how. If no, then give an example of a number d for which you can't do this.

d	n	$a(n)$
$\frac{1}{10}$		
$\frac{1}{100}$		
$\frac{1}{1{,}000}$		
$\frac{1}{10{,}000}$		

Next, consider the sequence

$$\left\{1, \frac{1}{2}, 1, \frac{1}{4}, 1, \frac{1}{6}, \dots\right\}$$

you can enter this as the following function

$$b(n) = \frac{1-(-1)^n}{2} + \frac{1+(-1)^n}{2n}$$

Repeat what you did with $a(n)$, that is try to find n so that $b(n)$ is less than $d = \frac{1}{10}, \frac{1}{100}, \frac{1}{1{,}000}$, and $\frac{1}{10{,}000}$. (Make a copy of this chart in your lab report.)

Question Q2. *Again, given any little d, can you always find n so that $b(n) < 0$? If you can't, give an example; if you can, explain how.*

Question Q3. *We said that $\lim_{n\to\infty} a(n) = 0$, would you also agree that $\lim_{n\to\infty} b(n) = 0$? (Why or Why Not?)*

d	n	$b(n)$
$\frac{1}{10}$		
$\frac{1}{100}$		
$\frac{1}{1{,}000}$		
$\frac{1}{10{,}000}$		

Let's look at each of these sequences a little differently. On separate graphs plot the first 10 terms in each sequence, $a(n)$ and $b(n)$, by plotting the points

$$(0, a(0)), (1, a(1)), (2, a(2)), \ldots, (19, a(9))$$

and

$$(1, b(1)), (2, b(2)), (3, b(3)), \ldots, (10, b(10))$$

finally add to each graph the horizontal line $y = 1/4$.

Question Q4. *What is the difference between the plot for $a(n)$ and for $b(n)$?*

Question Q5. *Does this change or confirm your conclusion about $\lim_{n\to\infty} b(n)$?*

For Part Ia of your lab report include the following:

1. Both tables,
2. Answer questions Q1, Q2, and Q3,
3. The graphs,
4. Answer questions Q4 and Q5,

Using the same sorts of methods as above decide whether each of these sequences has a limit or not. Be sure to give good supporting evidence for your claim and don't be afraid to experiment with different values of n and with plots.

1. $c(n) = 3 - \frac{1}{2^n}$: $\left\{2, 2\frac{1}{2}, 2\frac{3}{4}, 2\frac{7}{8}, 2\frac{15}{16}, \cdots\right\}$
2. $d(n) = 1 + \frac{(-1)^n}{n}$: $\left\{0, \frac{3}{2}, \frac{2}{3}, \frac{5}{4}, \frac{4}{5}, \cdots\right\}$
3. $e(n) = (-1)^n + \frac{1}{n}$: $\left\{0, \frac{3}{2}, -\frac{2}{3}, \frac{5}{4}, -\frac{4}{5}, \cdots\right\}$

For Part Ib of your lab report, include the following. For **each** of 1, 2, and 3 include a table and a graph and include a discussion which answers the following variation of Q1 for $c(n)$ (and likewise for $d(n)$ and $e(n)$).

Question Q1′. *We are looking for a number L with the following property. For any little d, can you always find an n so that $|c(n) - L| < d$? If yes, then try to describe why and how. If no, then give an example of a number d for which you can't do this.*
