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Chapter 1

The Three Pictures of Trigonometry

There are Three Pictures of Trigonometry. Each picture captures a different aspect of trigonometry. The pictures are associated with the following topics.

1. Measuring angles
2. The definition of sine and cosine
3. The graphs of sine and cosine

1.1.1 Angles and their measurement - Definitions

Picture # 1 is associated with the measurement of angles.
Angles are measured as a fraction of the full circle.

Figure 1.1: The measure of an angle

We have three examples. In the first and most familiar, the convention is that the full circle consists of 360° (degrees). If the blue arc represents 1/6-th of the full circle, then the measure of the blue arc in degrees is 1/6 th of 360°, or

\[ l = 360° \cdot \frac{1}{6} = 60°. \]

The measure of the full circle is up to the civilization that is measuring it. For example, I, leader of my new kingdom, choose to call the unit of angle measurement in my kingdom the zap. There are 51 zaps in a full circle. From then on, having made that choice, the size of an angle is measured in units of zaps. So the above angle is 1/6th of 51 zaps, or

\[ l = 51 \text{ zaps} \cdot \frac{1}{6} = 8.5 \text{ zaps}. \]

This is a rather unnatural unit of angle measurement, but would work perfectly well.

A relatively natural unit of measurement assigns the full circle a size, or measure, of 2π, the circumference of a circle with unit radius. The unit for this measure of an angle is called the radian after the radius. In this case, the above blue angle (θ) would be 1/6th of 2π, or

\[ l = 2\pi \text{ radians} \cdot \frac{1}{6} = \frac{\pi}{3} \text{ radians}. \]
1.1. THE THREE PICTURES OF TRIGONOMETRY

1.1.2 Conversions

We have just measured $l$ the length of the blue arc in three different units of measurement. If I have found $l$ in one unit, there is an easy way to convert to the value of $l$ in a different unit of measure. The conversion method is based on the fact that angles are measured as fractions of the full circle. So,

\[
\frac{\text{arc length}}{\text{full circle}} = \frac{l (\text{degrees})}{\text{full circle in (degrees)}} = \frac{l (\text{zaps})}{\text{full circle in (zaps)}} = \frac{l (\text{radians})}{\text{full circle in (radians)}}.
\]

Example 1 Suppose we have an angle with an arc that measures 67 degrees.

1. How many zaps would that angle measure?
2. How many radians would that angle measure?

Solution to Part 1. We set up the following equality, where $x$ is the unknown number of zaps.

\[
\frac{67^\circ}{360^\circ} = \frac{x (\text{zaps})}{51 (\text{zaps})}
\]

Solving for $x$ we get

\[
x (\text{zaps}) = \frac{67^\circ}{360^\circ} \cdot 51 (\text{zaps})
\]

or

\[
x (\text{zaps}) = \frac{51 (\text{zaps})}{360^\circ} \cdot 67^\circ.
\]

Staring at this for a bit we realize that it will always work. If we start with an angle measuring $y (\text{degrees})$ and we want the convert that measurement to a measurement in the units of zaps, we do the following.

\[
x (\text{zaps}) = \frac{51 (\text{zaps})}{360^\circ} \cdot y (\text{degrees}).
\]
Solution to Part 2. We do exactly the same process, only the units and the numbers change, the process remains the same. We set up the following equality, where $x$ is the unknown number of radians.

$$\frac{67^\circ}{360^\circ} = \frac{x \text{ (radians)}}{2\pi \text{ (radians)}}$$

Solving for $x$ we get

$$x \text{ (radians)} = \frac{67^\circ}{360^\circ} \cdot 2\pi \text{ (radians)}$$

or

$$x \text{ (radians)} = \frac{2\pi \text{ (radians)}}{360^\circ} \cdot 67^\circ.$$  

Staring at this for a bit we realize that it will also always work. If we start with an angle measuring $y \text{ (degrees)}$ and we want the convert that measurement to a measurement in the units of radians, we do the following.

$$x \text{ (radians)} = \frac{2\pi \text{ (radians)}}{360^\circ} \cdot y \text{ (degrees)}.$$  

Some of you have probably memorized this formula before. Me, I can’t remember it. I always forget it or get it upside-down. But I can remember the process, and thus don’t need to remember the formula. That said, when it comes times to take a quiz or an exam, investing a little time to memorize the formula will probably save a little time on the quiz or exam. But if you forget, well you can get what you need by equating the ratios, as we did here.

Figure 1.2: The meaning of radian measure of an angle
1.1. THE THREE PICTURES OF TRIGONOMETRY

Radian measure of angle

There is another aspect to the radian measure of angle. In Figure 1.2, if you measure the length of the blue arc; call that $a$, for arc length. And, if you measure the length of the radius; call that $r$, for radius. Then

$$\theta \text{ (radians)} = \frac{a}{r},$$

no matter what units of length you use to measure the arc length and the radius. In other words, measured in radians, the angle $\theta$ is measuring the length of the arc using the radius length to define the unit length.

1.1.3 The definition of sine and cosine

Picture # 2 is associated with the definition of sine.

![Figure 1.3: The definition of sine and cosine](image)

When we want to answer the question, what is the sine of an angle $\theta$, we use Figure 1.3. To describe what $\sin(\theta)$ is, one finds the point where the ray corresponding the angle $\theta$ intersects the unit circle. On that intersection point one puts a big red dot. Then drawing a horizontal red line to the $y$-axis, we obtain a specific height, the $y$ value where the dotted red line hits the $y$-axis. That height is $\sin(\theta)$. Notice this is also the height of the red line segment in Figure 1.3. The $\cos(\theta)$ is found by drawing a
green vertical line down to the $x$-axis. The value of $x$ at which the green line intersects the $x$-axis is the $\cos(\theta)$. The $\cos(\theta)$ is also the length of the green line segment in Figure 1.3.

That’s it.

This doesn’t really tell us how to find the values of sine and cosine, it just tells us what the numbers mean. If you were really, really good with a pencil, you could draw a really careful circle and a really careful 60° arc, and a really careful red dot and a really careful horizontal line, and really carefully measure the height of the red line, and you would have the $\sin(60^\circ)$. Similarly for $\cos(60^\circ)$. Try it.

In class we used a 45° triangle and an equilateral triangle to work out the values of $\sin(\theta)$ for $\theta$ equal to 30°, 60°, and 90°. In addition, we used Figure 1.3 to work out $\sin(\theta)$ for $\theta$ equal to 0 and 90°. We then used symmetry to extend this information to those values of $\theta$ between 90° and 180°. When we were done, we had the following table of data.

<table>
<thead>
<tr>
<th>$\theta^\circ$</th>
<th>0</th>
<th>30</th>
<th>45</th>
<th>60</th>
<th>90</th>
<th>120</th>
<th>135</th>
<th>150</th>
<th>180</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta$ (rad)</td>
<td>0</td>
<td>$\frac{\pi}{6}$</td>
<td>$\frac{\pi}{4}$</td>
<td>$\frac{\pi}{3}$</td>
<td>$\frac{\pi}{2}$</td>
<td>$\frac{2\pi}{3}$</td>
<td>$\frac{3\pi}{4}$</td>
<td>$\frac{5\pi}{6}$</td>
<td>$\pi$</td>
</tr>
<tr>
<td>$\sin(\theta)$</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{\sqrt{2}}$</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>1</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>$\frac{1}{\sqrt{2}}$</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
</tr>
</tbody>
</table>

By using symmetry we can extend this to the angles between 180° and 360° to get the following table of data.

<table>
<thead>
<tr>
<th>$\theta^\circ$</th>
<th>180</th>
<th>210</th>
<th>225</th>
<th>240</th>
<th>270</th>
<th>300</th>
<th>315</th>
<th>330</th>
<th>360</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta$ (rad)</td>
<td>$\pi$</td>
<td>$\frac{7\pi}{6}$</td>
<td>$\frac{5\pi}{4}$</td>
<td>$\frac{4\pi}{3}$</td>
<td>$\frac{3\pi}{2}$</td>
<td>$\frac{5\pi}{3}$</td>
<td>$\frac{7\pi}{4}$</td>
<td>$\frac{11\pi}{6}$</td>
<td>$2\pi$</td>
</tr>
<tr>
<td>$\sin(\theta)$</td>
<td>0</td>
<td>$-\frac{1}{2}$</td>
<td>$-\frac{1}{\sqrt{2}}$</td>
<td>$-\frac{\sqrt{3}}{2}$</td>
<td>$-1$</td>
<td>$-\frac{\sqrt{3}}{2}$</td>
<td>$-\frac{1}{\sqrt{2}}$</td>
<td>$-\frac{1}{2}$</td>
<td>0</td>
</tr>
</tbody>
</table>

Finally, using the same triangles and the same ideas of symmetry we can arrive
at a similar table of data for the cosine function.

\[
\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}
\hline
\theta^o & 0 & 30 & 45 & 60 & 90 & 120 & 135 & 150 & 180 & 210 & 225 & 240 & 270 & 300 & 315 & 330 & 360 \\
\hline
\theta (rad) & 0 & \frac{\pi}{6} & \frac{\pi}{4} & \frac{\pi}{3} & \frac{\pi}{2} & \frac{2\pi}{3} & \frac{3\pi}{4} & \frac{5\pi}{6} & \pi & \frac{7\pi}{6} & \frac{5\pi}{4} & \frac{4\pi}{3} & \frac{3\pi}{2} & \frac{5\pi}{3} & \frac{7\pi}{4} & \frac{11\pi}{6} & 2\pi \\
\hline
\cos(\theta) & 1 & \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{\sqrt{2}} & -\frac{\sqrt{3}}{2} & -1 & -\frac{\sqrt{3}}{2} & -\frac{1}{\sqrt{2}} & -\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{\sqrt{3}}{2} & 1 \\
\hline
\end{array}
\]

OH MY GOD! No, really it’s not that bad. You don’t have to memorize it. Just know how to use the triangles to construct the value, as we did in class.
CHAPTER 1. THE THREE PICTURES OF TRIGONOMETRY

Problems.
1. For the angle $\theta$ depicted in the figure to the right, what is the exact value of $\sin(\theta)$?

2. Convert $137^\circ$ to radians.

3. Convert $247^\circ$ to radians (Round to 2 decimal places).

4. A DVD rotates $5/6$ of a revolution. How many radians has it travelled?

5. In a circle with radius of 6 yards find the radian measure of the central angle whose terminal side intersects the circle forming an arc which measures 8 yards in length.

6. What are the exact values of the following.

   a. $\cos(30^\circ) =$
   b. $\sin(45^\circ) =$
   c. $\cos(60^\circ) =$
   d. $\sin(30^\circ) =$
   e. $\cos(270^\circ) =$
   f. $\sin(90^\circ) =$
   g. $\cos(150^\circ) =$
   h. $\sin(135^\circ) =$
   i. $\cos(225^\circ) =$
   j. $\sin(330^\circ) =$
   k. $\cos(270^\circ) =$
   l. $\sin(150^\circ) =$
   m. $\cos(\frac{\pi}{4}) =$
   n. $\sin(\frac{5\pi}{4}) =$
   o. $\cos(\frac{3\pi}{4}) =$
   p. $\sin(\frac{5\pi}{6}) =$
   q. $\cos(\frac{11\pi}{6}) =$
   r. $\sin(2\pi) =$
   s. $\cos(\pi) =$
1.1.4 Tangent, Cotangent, Secant, and Cosecant

We also define four more functions directly from the graph of the unit circle just like we did for sine and cosine. See Figure 1.4.

Figure 1.4: The definition of tangent, cotangent, secant and cosecant

\[
\begin{align*}
\tan(\theta) &= \frac{y}{x} \\
\cot(\theta) &= \frac{x}{y} \\
\sec(\theta) &= \frac{1}{x} \\
\csc(\theta) &= \frac{1}{y}
\end{align*}
\]

These functions can be related to the values of sine and cosine as follows.

\[
\begin{align*}
\tan(\theta) &= \frac{\sin(\theta)}{\cos(\theta)} \\
\cot(\theta) &= \frac{\cos(\theta)}{\sin(\theta)} \\
\sec(\theta) &= \frac{1}{\cos(\theta)} \\
\csc(\theta) &= \frac{1}{\sin(\theta)}
\end{align*}
\]

These relationships are typically used to determine the values of these functions and to understand their properties.

**Example 2** Find the exact value of \(\tan(\pi/4)\).

**Answer:** Because \(\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}\) we have

\[
\tan(\pi/4) = \frac{\sin(\pi/4)}{\cos(\pi/4)} = \frac{1}{\sqrt{2}} = 1.
\]
Problems.
What are the exact values of the following.

1. \( \tan(30^\circ) = \)
2. \( \sec(30^\circ) = \)
3. \( \sec(135^\circ) = \)
4. \( \tan(120^\circ) \)
5. \( \tan(240^\circ) = \)
6. \( \csc(240^\circ) = \)
7. \( \tan\left(\frac{2\pi}{3}\right) = \)
8. \( \cot\left(\frac{3\pi}{4}\right) = \)
9. \( \sec\left(\frac{2\pi}{3}\right) = \)
10. \( \cot\left(\frac{4\pi}{3}\right) = \)
1.2 The Graphs of Sine and Cosine

1.2.1 Expansions, Contractions and Shifts

Our goal is now to explore the third picture of trigonometry, the graph of \( \sin(x) \). In addition to examining the graph of \( \sin(x) \), we will explore how we can get new graphs from the graph of \( \sin(x) \) by expansions, contractions, and shifts. One motivation is the pendulum.

In class we may have measured the position of a pendulum with our tricorder. It would have displayed data like that in Figure 1.6.
We would like to model this data with a $\sin(x)$ function using some combination of expansion, contraction, and or shifting. To begin we examine the graph of the function $\sin(x)$. Earlier we constructed a table of data on the sine function. Figure 1.7 depicts a graph of that data.

If we struggle to get the values for more points (and how might we do that?), we can extend the graph to get a shape that seems to look more and more like that in Figure 1.8. When we address the derivative of $\sin(x)$ and $\cos(x)$ we will see further confirmation that the shape in Figure 1.8 is the correct graph.
1.2. THE GRAPHS OF SINE AND COSINE

Figure 1.8: The Graph of $\sin(x)$

1.2.2 The slope of $\sin(x)$ at $x = 0$.

In class we may have discussed further evidence that Figure 1.8 is the correct shape for the graph by exploring what we expected the slope of the graph to be at $x = 0$, that is, when the angle is zero. We would have done this by examining the diagram used to define $\sin(x)$. In the right hand side of Figure 1.9 we see that as the angle $x$

Figure 1.9: Considering $\lim_{x \to 0} \frac{\sin(x)}{x}$

get small, the ratio of $\sin(x)$ (the length of the red, vertical line segment) to $x$ (angle measured in radians, the blue arclength) is getting closer and closer to 1 - because the two lengths are getting closer and closer. (It is true, both are getting smaller and
smaller, but their lengths are getting closer even faster.) This confirms what we see in Figure 1.8, where the slope of the graph of $\sin(x)$ at zero is positive. If we were to rescale the figure, so that 1 unit in the $x$ direction and 1 unit in the $y$ direction were the same size, we would see the slope would actually look like it equaled 1.

1.2.3 The midline

When we discussed the midline, the center of $\sin(x)$ we saw that the function

$$f(x) = A + \sin(x)$$

has its midline lifted $A$ units, which means down if $A$ is negative. See Figure 1.10. We say the graph of $f(x) = A + \sin(x)$ is a vertical shift of the graph of $\sin(x)$.

Figure 1.10: $\sin(x)$ versus $A + \sin(x)$

1.2.4 The amplitude

In contrast, there is amplitude which measures the distance between the midline and the extremes, the maximum and minimum. Compare the graphs of $\sin(x)$ and $B \sin(x)$ as seen in Figure 1.11. As we discussed in class the amplitude of $\sin(x)$
1.2. THE GRAPHS OF SINE AND COSINE

is 1, and the amplitude of $B \sin(x)$ is $|B|$ (the absolute value, because $B$ could be negative). If $|B| > 1$, then $B \sin(x)$ is called a vertical expansion of $\sin(x)$. If $|B| < 1$, then $B \sin(x)$ is called a vertical contraction of $\sin(x)$.

Figure 1.11: $\sin(x)$ versus $B \sin(x)$
1.2.5 Frequency and period.

Those were vertical changes, let us move to horizontal changes. Consider the function $h(x) = \sin(Cx)$. Let’s compare $h(x)$ with $\sin(x)$. See Figure 1.12. Is it clear what is going on? No? Let us try a few simpler cases. Let us try $C = 2$ and $C = 3$, as shown in Figure 1.13. After staring for a while, what we notice is that $\sin(2x)$ goes through two cycles while $\sin(x)$ only goes through one. And $\sin(3x)$ goes through three cycles while $\sin(x)$ only goes through one. One year in class Arron referred to this as the frequency. The frequency of $\sin(2x)$ has doubled, and the frequency of $\sin(3x)$ has tripled.
1.2. THE GRAPHS OF SINE AND COSINE

The frequency is related to the period. People will often use period (instead of frequency) because the period’s value can be read right off the graph. Measure the distance between two adjacent peaks; that distance is the period. Measure the distance between two adjacent valleys, and you get the same value, the period.

The period is the distance it takes a function to go through one cycle. How does one express that mathematically? It’s a tad tricky (it takes two steps). Why two? Well, not all functions have periods, say for example $f(x) = x^2$ - it has no period.

First we have to determine what functions can have a period. We say, a function $f(x)$ is periodic if there exists a positive (non-zero) number $c$ such that

$$f(x + c) = f(x). \quad (1.5)$$

(For $f(x) = \sin(x)$, $c = 2\pi$, or $c = 4\pi$, or $c = 100\pi$, and any other multiple of $\pi$ works.) Now that we know what function can have periods (the periodic ones), we can address what that period is. The period of a periodic function is the smallest positive (non-zero) value $c$ such that

$$f(x + c) = f(x) \quad (1.6)$$

holds.

Examining Figure 1.13 we see that value of $c$ is $c = 2\pi$ in the sense that

$$\sin(x + 2\pi) = \sin(x).$$

Obvious? Maybe not. There is another way of looking at this. Let us return to the picture we used to define the sine function. Let’s pick a specific angle $\theta$ to work with, say $\theta = \pi/4$. Let’s be clear about the following. When we write $\theta \neq \theta + 2\pi$ we
simply mean something like the following

\[ \frac{\pi}{4} \neq \frac{\pi}{4} + 2\pi. \]

The left hand side is less than 1, the right hand side is getting close to 7; they can’t be equal.

But, you say, the angle looks the same. **That is true!** However, consider the following. Grab ahold of a long piece of string. I’ll hold the other end, and you spin around twice. That is, holding onto the string spin through an angle of \(4\pi\). You have the string wrapped around you twice. Repeat the procedure but only rotate once. That is, grab ahold of the string (after disentangling youself from the first attempt) and rotate through an angle of \(2\pi\) (one rotation) you find yourself with the string only wrapped around you once. At the end of both procedures you are facing the same way and everything seems the same, which it is if you are not holding onto any string. But everything is not the same if you are holding onto a piece of string. The measure of the angle (\(4\pi\) versus \(2\pi\)) distinguishes the two circumstances, having the string wrapped around you twice versus having the string wrapped around you once.

Now, let us discuss the effect of an additional rotation or two on the sine function.
Examine Figure 1.14 we see the \( \sin(\theta) \) is only concerned with the height of the red line segment, and doesn’t care about any piece of string or what had to be done to get to where we are. The \( \sin(\theta) \) is only concerned with where the ray at an angle \( \theta \) is pointing now. If we go another rotation to get to the angle \( \pi/4 + 2\pi \), the ray points in the same direction and the height of the red line segment now \( \sin(\pi/4 + 2\pi) \) is the same as it was before the additional rotation, \( \sin(\pi/4) \). Which is to say

\[
\sin(\pi/4 + 2\pi) = \sin(\pi/4)
\]

and more generally

\[
\sin(\theta + 2\pi) = \sin(\theta).
\]

Likewise we have

\[
\cos(\theta + 2\pi) = \cos(\theta) \quad \text{and} \quad \tan(\theta + 2\pi) = \tan(\theta)
\]

\[
\cot(\theta + 2\pi) = \cot(\theta) \quad \text{and} \quad \sec(\theta + 2\pi) = \sec(\theta) \quad \text{and} \quad \csc(\theta + 2\pi) = \csc(\theta).
\]

The period of \( \sin(x) \), \( \cos(x) \), \( \sec(x) \), and \( \csc(x) \) is \( 2\pi \), however, \textbf{note}: the period of \( \tan(x) \) and \( \cot(x) \) is just \( \pi \)!

1.2.6 The period of \( \sin(Cx) \)

Let \( h_C(x) = \sin(Cx) \) and \( p_C \) refer to the period of \( h(x) \). Let \( p_s \) refer to the period of \( \sin(x) \). That is, \( p = 2\pi \).

Let’s return to our conversation about frequency, and refer to Figure 1.13, copied here as Figure 1.15. Look at Figure 1.15. It was observed that the frequency of \( h_2(x) = \sin(2x) \) was twice that of \( \sin(x) \). So, \( h_2(x) = \sin(2x) \) goes through two cycles while \( \sin(x) \) only goes through one cycle. Since \( p_2 \) is the distance it takes for \( h_2(x) \)
Figure 1.15: sin($x$) versus sin(2$x$) and sin(3$x$)

to go through a cycle, and $h_2(x)$ goes through two cycles, it must cover a distance of $2p_2$ doing so. sin($x$) only goes through one cycle in the same distance of $p = 2\pi$. $2 \cdot p_2$ and $p = 2\pi$ are the same distance

$$2 \cdot p_2 = p = 2\pi.$$ 

We can solve for the period $p_2$ of $h_2(x) = \sin(2x)$. We get $p_2 = \pi$.

In a similar fashion we can solve for the period of $h_3(x) = \sin(3x)$ using the same reasoning to end up with

$$3 \cdot p_3 = p = 2\pi.$$ 

Solving for the period $p_3$ of $h_3(x) = \sin(3x)$, we get $p_3 = \frac{2\pi}{3}$.

What do we do for the general situation, $h_C(x) = \sin(Cx)$? We use the same approach, which with the same careful reasoning we can argue through to arrive at the same relation

$$C \cdot p_C = p = 2\pi.$$ 

Solving for the period $p_C$ of $h_C(x) = \sin(Cx)$, we get $p_C = \frac{2\pi}{C}$. The careful reasoning referred to above is the same sort of reasoning on might use to build the rationals
and the real numbers out of the natural numbers. If you don’t want to embark on this little project, you could just take my word for it.

**Example 3** What is the period of \( f(x) = \sin(7x) \)?

**Solution.** Let \( 7p = 2\pi \) where \( p \) is the period. Solving for \( p \) we get

\[
p = \frac{2\pi}{7}.
\]

**Example 4** What is the period of \( f(x) = \cos(11.4x) \)?

**Solution.** Let \( 11.4p = 2\pi \) where \( p \) is the period. Solving for \( p \) we get

\[
p = \frac{2\pi}{11.4}.
\]

**Example 5** What is the period of \( f(x) = 5 + 3\sin\left(\frac{2\pi x}{3}\right) \)?

**Solution.** Let \( \frac{2\pi}{3}p = 2\pi \) where \( p \) is the period. Solving for \( p \) we get

\[
p = \frac{2\pi}{\frac{2\pi}{3}} = \frac{2\pi}{1} \cdot \frac{3}{2\pi} = 3.
\]

**Example 6** What is the period of \( f(x) = \pi + 2\sin\left(\frac{3(x-1)}{4}\right) \)?

**Solution.** Multiply the portion inside sine function, that is multiply out \( \frac{3(x-1)}{4} \). This gives us \( \frac{3(x-1)}{4} = \frac{3x}{4} - \frac{3}{4} \). Taking the term with the \( x \) (the \( \frac{3}{4} \)), let \( \frac{3}{4}p = 2\pi \) where \( p \) is the period. Solving for \( p \) we get

\[
p = \frac{2\pi}{\frac{3}{4}} = \frac{8}{3}\pi.
\]
Again, Note: The period of the tangent and cotangent functions are a little different. Their period is $\pi$, not $2\pi$.

1.2.7 Horizontal shifts

Finally we discuss horizontal shifts. These work the same for sine and cosine as they do for polynomials. Let’s remind ourselves how this works for polynomials, and to do so let’s see how it works for our favorite polynomial, $f(x) = x^2$. If we compare the graphs of

$$f(x) = x^2 \text{ and } g(x) = (x - 3)^2$$

what do we see? See Figure 1.16. Ah, yes the graph of $x^2$ is shifted right by 3. We learned when $p(x)$ was a polynomial, that the new function $g(x) = p(x - a)$ had a graph that looked like the graph of $p(x)$ shifted right by the amount $a$. If $a$ was negative, then that means shifted left.

For trigonometric functions, nothing has changed.

**Example 7** Let $f(x) = \sin(x - \frac{\pi}{4})$ how far must the graph of $\sin(x)$ be shifted to produce the graph of $f(x)$.
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Solution. Regarding \( f(x) = \sin(x - a) \) we see that \( a = \frac{\pi}{4} \), thus the shift is \( \frac{\pi}{4} \).

Example 8 Let \( f(x) = \sin(x - \frac{1}{4}) \) how far must the graph of \( \sin(x) \) be shifted to produce the graph of \( f(x) \).

Solution. Regarding \( f(x) = \sin(x - a) \) we see that \( a = \frac{1}{4} \), thus the shift is \( \frac{1}{4} \).

Compare the previous two examples carefully!

What if \( x \) has a coefficient? Then you must factor.

Example 9 Let \( f(x) = \sin(2x - \frac{\pi}{4}) \). How far must the graph of \( \sin(2x) \) be shifted to produce the graph of \( f(x) \)?

Solution. To see how much of a shift there is we must regard \( f(x) = \sin(2(x - a)) \).

To do this we must factor \( 2x - \frac{\pi}{4} = 2 \left( x - \frac{\pi}{8} \right) \). We see that

\[
f(x) = \sin(2(x - a)) = \sin(2(x - \frac{\pi}{8}))
\]

and thus \( a = \frac{\pi}{8} \). So the graph of \( f(x) = \sin(2x - \frac{\pi}{4}) \) is the graph of \( \sin(2x) \) the shifted right by \( \frac{\pi}{8} \). Let’s take a look at a graph of this; see Figure 1.17. Notice that the graph

Figure 1.17: The graph of \( \sin(2x) \) and \( \sin(2x - \frac{\pi}{4}) \)

of \( \sin(2x - \frac{\pi}{4}) \) is indeed the graph of \( \sin(2x) \) shifted by \( \frac{\pi}{8} \) and not the \( \frac{\pi}{4} \) that it may have at first seemed it should be.
Example 10  Let \( f(x) = 2 + 3 \sin(3x - \frac{\pi}{4}) \) how far must the graph of \( 2 + 3 \sin(3x) \) be shifted to produce the graph of \( f(x) \)?

Solution. Regarding \( f(x) = 2 + 3 \sin(3(x - a)) \) and factoring \( 3x - \frac{\pi}{4} = 3(x - \frac{\pi}{12}) \) we see that \( a = \frac{\pi}{12} \), thus the shift is \( \frac{\pi}{12} \). It would be a good exercise to graph these two function on your calculator or on Derive to see that this is the correct shift.

Problems.

Problem 11  Let \( f(x) = \sin(2x - \frac{\pi}{4}) \). How far must the graph of \( \sin(2x) \) be shifted to produce the graph of \( f(x) \)?

Problem 12  Let \( f(x) = \sin(3x - \frac{2\pi}{7}) \). How far must the graph of \( \sin(3x) \) be shifted to produce the graph of \( f(x) \)?

Problem 13  Let \( f(x) = \sin(\frac{2x}{3} - 2) \). How far must the graph of \( \sin(2x) \) be shifted to produce the graph of \( f(x) \)?

Problem 14  Let \( f(x) = 2 + 3 \sin(3x - \frac{\pi}{4}) \) how far must the graph of \( 2 + 3 \sin(3x) \) be shifted to produce the graph of \( f(x) \)?

Problem 15  What is the midline, amplitude, and period of the following functions?

a. \( f(x) = \sin(3x) \)

b. \( f(x) = 5 \sin(4x) \)

c. \( f(x) = 7 \sin(3\pi x) \)

d. \( 5 + 3 \sin(\frac{\pi x}{4}) \)

e. \( 2 + \sin(\frac{x}{3}) \)

f. \( \cos(2x) \)
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\[
g. \ -\frac{3}{2} \cos\left(\frac{6\pi}{5}\right) \\
h. \ 4 \cos(2\pi x) \\
i. \ 2 \cos\left(\frac{x}{2}\right)
\]

**Problem 16** How far must the graph of \( \sin(x) \) be shifted to produce the graph of \( f(x) \), where \( f(x) \) is the following.

a. \( f(x) = \sin(x - \frac{2\pi}{3}) \)

b. \( f(x) = \sin(x + \frac{\pi}{2}) \)

c. \( f(x) = \sin(x - \frac{1}{5}) \)

d. \( f(x) = \sin(x - 2) \)

e. \( f(x) = \sin(x - 3\pi) \)
1.3 Geometric Applications

1.3.1 Triangle problems

A traditional problem has a building of unknown height. The problem is to determine the height. We do so by standing a known distance away from the building and sighting up at the top of the building, and measuring (what is known as) the angle of inclination from the ground up to the top of the building. See Figure 1.18.

Example 17  Suppose that the angle of inclination to the top of a building is found to be $35^\circ$ when measured at a distance of 40 feet from the base of a building. How tall is the building?

Figure 1.18: A Standard Triangle Problem

To answer this problem, one usually resorts to the triangle definition of the trigonometric functions. The triangle definitions are a way of understanding the sine, cosine, and tangent of an angle that only makes sense for some, but not all, angles. The triangle method lacks a certain elegance and generality that eventually makes the triangle definition difficult to work with. In order to use triangles to define sine and cosine for all angles one must construct up to four different variants of the definition;
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for this reason it is not as appealing as the circle definition in the long run. Some
of you may have learned this definition in high school. We now discuss the triangle
definition of sine and cosine.

1.3.2 Triangle Definitions of Trigonometric Functions

Consider a right triangle with sides $a$, $b$, and $c$, and the angle $\theta$. The side $a$
is sometimes known as the adjacent side because it is adjacent to the angle. The side $b$
is known as the opposite side because it is opposite to the angle. The side $c$ is called
the hypotenuse. See Figure 1.19.

Figure 1.19: The Triangle Definition of Sine, Cosine, and Tangent

\[ \sin(\theta) = \frac{\text{length of } b}{\text{length of } c} \]  \hspace{1cm} (1.7)

We can check that this definition is consistent with the circle definition.
In a similar fashion we define cosine and tangent.

\[ \cos(\theta) = \frac{\text{length of } a}{\text{length of } c} \quad (1.8) \]

\[ \tan(\theta) = \frac{\text{length of } b}{\text{length of } a} \quad (1.9) \]

**Solution to Example 22:** To solve this problem the quick way, we compare Figure 1.19 to Figure 1.18, and observe that what we are looking for is side \( b \) in Figure 1.19. That means we can take the definition of tangent and cross multiply by the length of \( b \). We get

\[ (\text{length of } a) \cdot \tan(35^\circ) = \text{length of } b. \]

or

\[ \text{length of } b = 40 \cdot \tan(35^\circ) \approx 40 \cdot 0.70 = 28. \]

We conclude the building height is 28 feet.

Some may wish to solve this problem via the intermediate step of first finding the length of the hypotenuse, \( c \). For some this approach offers the advantage that they can avoid working with the tangent function. I do not endorse this, nonetheless for this method we use the definition of cosine to first find the hypotenuse by cross multiplying as follows.

\[ (\text{length of } c) \cdot \cos(35^\circ) = \text{length of } a. \]

Dividing both sides by the cosine we get

\[ \text{length of } c = \frac{\text{length of } a}{\cos(35^\circ)}. \]

or

\[ \text{length of } c \approx \frac{40}{0.819} \approx 48.8. \]
Then we use the definition of sine and the length of $c$ to find the length of $b$.

$$\text{length of } b = (\text{length of } c) \cdot \sin(35^\circ) \simeq 48.8 \cdot 0.574 \simeq 27.99$$

We get essentially the answer we had before, the difference due to the number of significant figures we kept in the intermediate calculations.

**Example 18** Suppose an unidentified flying object is sighted at a height of 20 miles at an angle of inclination of $30^\circ$. Ten minutes later it has an angle of inclination of $35^\circ$. Assuming it is moving at a level altitude and at a constant speed straight toward the observer, how fast is the object moving?

**Solution to Example 18:** We begin by drawing a picture. There are two triangles in Figure 1.20, one with dotted lines and one with solid lines. Our approach will be to find the length of the base of each. We will use the tangent function.

$$\tan(\theta) = \frac{20}{\text{base}}$$

Cross multiplying by the base we get

$$\text{base} \cdot \tan(\theta) = 20$$
and dividing both sides by the tangent we get

\[ \text{base} = \frac{20}{\tan(\theta)}. \]

There are two bases, the dotted base and the solid base. The distance travelled in 10 minutes is their difference. We get

\[ (\text{solid base}) - (\text{dotted base}) = 20\left(\frac{1}{\tan(30)} - \frac{1}{\tan(35)}\right). \]

Because \( \tan(30) = \frac{1}{\sqrt{3}} \) and \( \tan(35) \approx 0.700 \) we get the distance travelled is \( d \approx 20(\frac{1}{\sqrt{3}} - 0.700) = 20(.303) = 6.07 \) miles. 6.07 miles in 10 minutes (1/6th of an hour) is 36.42 miles per hour. Hmm, is our UFO a weather balloon?

An additional comment. Because the \( \frac{1}{\tan(\theta)} \) expression can occur frequently, the cotangent function is defined as follows.

\[ \cot(\theta) = \frac{1}{\tan(\theta)}. \]

Then the above distance formula looks like this.

\[ (\text{solid base}) - (\text{dotted base}) = 20\left(\frac{1}{\tan(30)} - \frac{1}{\tan(35)}\right) = 20 (\cot(30) - \cot(35)). \]

The last expression (on the right) being regarded as simpler, and easier on the eyes.

Problems.

Problem 19  For the following items refer to Figure 1.21.

a. Suppose \( a = 10 \) and \( \theta = 20^\circ \). Find \( b \) and \( c \).

b. Suppose \( a = 15 \) and \( \theta = 75^\circ \). Find \( b \) and \( c \).

c. Suppose \( b = 12 \) and \( \theta = 30^\circ \). Find \( a \) and \( c \).
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Figure 1.21: Problem 1

![Right Triangle Diagram]

d. Suppose $b = 50$ and $\theta = 20^\circ$. Find $a$ and $c$.

e. Suppose $c = 88$ and $\theta = 85^\circ$. Find $a$ and $b$.

f. Suppose $c = 14$ and $\theta = 45^\circ$. Find $a$ and $b$.

g. Suppose $a = 10$ and $b = 5$. Find $c$ and $\theta$.

h. Suppose $a = 10$ and $c = 15$. Find $b$ and $\theta$.

**Problem 20** Suppose that the angle of inclination to the top of a building is found to be $38^\circ$ when measured at a distance of 52 feet from the base of a building. How tall is the building?

**Problem 21** Suppose that the angle of inclination to the top of a building is found to be $72^\circ$ when measured at a distance of 38 feet from the base of a building. How tall is the building?

**Problem 22** Suppose that the angle of inclination to the top of a mountain is found to be $4.5^\circ$ when measured at a distance of 6.5 miles from what one estimates to be the base of the peak. How tall is the mountain in feet?
Problem 23 Suppose there is a sign on the top of the building, as in Figure 1.22. Suppose the angle of inclination to the top of the building is 40° and suppose the angle of inclination to the top of the sign is 44°. If these angle measurements are taken at a distance of 50 feet from the building, how high is the sign?

Problem 24 Suppose there is a sign on the top of the building, as in Figure 1.22. Suppose the angle of inclination to the top of the building is 47° and suppose the angle of inclination to the top of the sign is 55°. If these angle measurements are taken at a distance of 75 feet from the building, how high is the sign?

1.3.3 The Law of Sines

The law of sines is the following relation between angles and side lengths, where the side and angle labels refer to Figure 1.23.

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c} \quad (1.10)$$

Example 25 Referring to Figure 1.23, suppose $a = 15$, $B = 42°$ and $A = 30°$. Determine the other angles and sides.
Solution. The expressions in Equation 1.10 involving \(a\), \(A\), \(b\), and \(B\) are the relevant ones. That is,

\[
\frac{\sin(A)}{a} = \frac{\sin(B)}{b}.
\]

Plugging in the relevant, known information, we get

\[
\frac{\sin(30^\circ)}{15} = \frac{\sin(42^\circ)}{b}.
\]

Rearranging to solve for \(b\), we have

\[
b = 15 \frac{\sin(42^\circ)}{\sin(30^\circ)} \approx 15 \frac{0.669}{0.5} \approx 20.1.
\]

To solve for the side \(c\) and the angle \(C\), we first use the fact that the angles in a triangle add up to 180° (or \(\pi\) when measured in radians), to solve for \(C\).

\[
30^\circ + 42^\circ + C = 180^\circ
\]

or

\[
C = 180 - 30 - 42 = 108.
\]

Now we proceed as we did above, when we were finding \(b\).

\[
\frac{\sin(C)}{c} = \frac{\sin(A)}{a}.
\]
Pluggin $a$, $A$, $C$ and rearranging we get

$$c = 15 \frac{\sin(108^\circ)}{\sin(30^\circ)} \simeq 15 \frac{0.951}{0.5} \simeq 28.5.$$ 

Our conclusion $b \approx 20.1$, $c \approx 28.5$, and $C = 108^\circ$.  

**Example 26** Referring to Figure 1.23, suppose $a = 14$, $A = 30^\circ$, and $b = 15$. Determine the other angles and sides. 

**Solution.** We first notice that we have two sides and one angle. This makes us suspicious. We draw a candidate triangle, a triangle consistent with our data (see Figure 1.24). Hmm, we need to be careful. It seems from the picture (Figure 1.24) that there are two possibilities.

The expressions in Equation 1.10 involving $a$, $A$, $b$, and $B$ are the relevant ones. As before, we copy

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b}.$$ 

Plugging in the relevant known information, we get

$$\frac{\sin(30^\circ)}{14} = \frac{\sin(B)}{15}.$$
Rearranging to solve for $\sin(B)$, we have

$$\sin(B) = \sin(30^\circ) \frac{15}{14} = \frac{15}{28} \approx 0.536.$$  

Solving for $B$ we get $B = \sin^{-1}(0.536) \approx 32.4^\circ$.

However, recalling Figure 1.24 (look at it!) we can see that there are two plausible angles. (We also recall from the circular definition of sine, that there are two angles which have a sine of 0.536, one is $32.4^\circ$ and the other is $180^\circ - 32.4^\circ = 147.6^\circ$.) We proceed.

**Case i.** $B = 32.4^\circ$. From here on out we use exactly the same approach as the previous problem. Refer to it. We solve for $C$; $C = 180 - 30 - 32.4 = 117.6$. Next we solve for $c$.

$$c = 14 \frac{\sin(117.6^\circ)}{\sin(30^\circ)} \approx 14 \frac{0.886}{0.5} \approx 24.8.$$  

**Case i Solution:** $A = 30^\circ$, $B = 32.4^\circ$, $C = 117.6^\circ$, $a = 14$, $b = 15$, $c = 24.8$.

**Case ii.** $B = 147.6^\circ$. We solve for $C$; $C = 180 - 30 - 147.6 = 2.4$. Next we solve for $c$.

$$c = 14 \frac{\sin(2.4^\circ)}{\sin(30^\circ)} \approx 14 \frac{0.0419}{0.5} \approx 1.2.$$  

**Case i Solution:** $A = 30^\circ$, $B = 147.6^\circ$, $C = 2.4^\circ$, $a = 14$, $b = 15$, $c = 1.2$.  

**Problem 27** For the following items refer to Figure 1.23.

a. Suppose $a = 10$, $B = 30^\circ$ and $A = 20^\circ$. Determine the other angles and sides.

b. Suppose $b = 18$, $B = 35^\circ$ and $A = 25^\circ$. Determine the other angles and sides.
c. Suppose $a = 10$, $b = 30$ and $A = 40^\circ$. Determine the other angles and sides.

d. Suppose $a = 10$, $b = 35^\circ$ and $C = 25^\circ$. Determine the other angles and sides.

**Problem 28** The residents of Lake Skatatoochie wish to know how long their lake is. There are two straight roads that intersect near Skatatoochie (see Figure 1.25). The distances from the intersection $I$ to point $A$ is 2. The angle of intersection of the roads ($\angle AIB$) is $109^\circ$ and the angle $\angle IAB = 45^\circ$. Roughly, how long is the lake?

![Figure 1.25: Lake Skatatoochie](image)

**Problem 29** Also on Lake Skatatoochie is a house located at $C$. Can you determine how far the house is from $A$ and from $B$? Would it help to know that the angle $\angle IAC = 66^\circ$?

1.3.4 Why is the Law of Sines what it is?
1.3.5 The Law of Cosines

The Law of Cosines is a generalization of the Pythagorean Theorem, or the equation therein, and looks like the following.

\[ c^2 = a^2 + b^2 - 2ab \cos(C) \]  

(1.11)

where \( C \) is the angle opposite the side \( c \). Refer to Figure 1.26 if it helps.

Example 30 Refer to Figure 1.26. Suppose we know \( a = 7.5 \), \( b = 5.5 \) and \( C = 42^\circ \). What are the other angles and what is the length of the other side? (We might note, we don’t have enough information to employ the Law Sines, yet. Why?)

Figure 1.26: Example 30

Solution. The label of the other side is \( c \), so we are in fact able to use the Law of Cosines just as it is written.

\[ c^2 = 7.5^2 + 5.5^2 - 2(7.5)(5.5) \cos(42) \]  

(1.12)

Or \( c^2 = 56.25 + 30.25 - 82.5 \cos(42) \approx 25.2 \). Thus \( c = \sqrt{25.2} \approx 5.02 \).

From here we can use the law of sines, or we can practice the law of cosines. We are going to practice the law of cosines. We will apply Equation 1.11 with the symbols permuted. We exchange the symbols \( a \) and \( c \) and change \( C \) to \( A \) to get.

\[ a^2 = c^2 + b^2 - 2cb \cos(A) \]  

(1.13)
We can solve this for $A$ as follows.

\[-2cb \cos(A) = a^2 - b^2 - c^2\]

Divide by the $-2cb$ to get

\[\cos(A) = \frac{a^2 - b^2 - c^2}{-2cb}.\]

Finally, take the $\cos^{-1}$ of both sides to get

\[A = \cos^{-1}\left(\frac{a^2 - b^2 - c^2}{-2cb}\right)\.

We know $a = 7.5$, $b = 5.5$, and $c \simeq 5.02$. So

\[A = \cos^{-1}\left(\frac{(7.5)^2 - (5.5)^2 - (5.02)^2}{-2(5.02)(5.5)}\right) \simeq 90.83^\circ\]

A similar calculation can be made by exchanging the symbols $b$ and $c$ in Equation 1.11. We have

\[b^2 = a^2 + c^2 - 2ac \cos(B)\] (1.14)

We can solve this for $B$ in the same fashion.

\[B = \cos^{-1}\left(\frac{b^2 - a^2 - c^2}{-2ca}\right)\]

We know $a = 7.5$, $b = 5.5$, and $c \simeq 5.02$. So

\[A = \cos^{-1}\left(\frac{(5.5)^2 - (7.5)^2 - (5.02)^2}{-2(5.02)(7.5)}\right) \simeq 47.16^\circ\]

We can check our answer by adding the angles up.

\[42 + 90.83 + 47.16 = 179.99\]

Well, it should be 180. Not bad, considering there will be a little error due to rounding.

In summary, $a = 7.5$, $b = 5.5$, $c = 5.02$, $A = 42^\circ$, $B = 47.16^\circ$, and $C = 90.83^\circ$. ■
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Problem 31  For the following items refer to Figure 1.23.

a. Suppose $a = 10$, $b = 10$ and $C = 30^\circ$. Determine the other angles and sides.

b. Suppose $b = 18$, $c = 25$ and $A = 45^\circ$. Determine the other angles and sides.

c. Suppose $a = 10$, $c = 30$ and $B = 40^\circ$. Determine the other angles and sides.

d. Suppose $a = 10$, $b = 35$ and $C = 75^\circ$. Determine the other angles and sides.

Problem 32  The residents of Lake Skatatoochachie wish to know how long their lake is. Like Skatatoochie, there are two straight roads that intersect near Skatatoochachi (see Figure 1.27). The distances from the intersection to point $A$ is 1.8 miles and the distance to point $B$ is 2.5 miles. The angle of intersection of the roads is $116^\circ$. Roughly, how long is the lake?

Figure 1.27: Lake Skatatoochachie

Problem 33  Also on Lake Skatatoochachie is a house located at $C$. Can you determine how far the house is from $A$ and from $B$? Would it help to know that the angles $\angle AIC$ and $\angle BIC$ are the same?

1.3.6  Why is the Law of Cosines what it is?
1.4 Trigonometric Equations

We want to practice solving expressions like the following.

$$2 \sin(x) = 1$$

When we say we want to solve this, we mean we want to find the values of $x$, numbers, which when plugged into the equation make it true.

There is a way to visualize this. The equation above has two parts, $f(x) = 2 \sin(x)$ on the left hand side and $g(x) = 1$ on the right hand side. In Figure 1.28, below, we see the graphs of these two functions.

The solutions to the equation above correspond to the graph intersections. More specifically, the solutions are numbers, values of $x$, at which the graphs intersect. In Figure 1.30 we have marked the points in the plane where the graphs intersect, and we have marked the numbers on the $x$ axis corresponding to those intersections. The numbers, the $x$ values, are the solutions we seek.
To find the solution there are two steps. First, we find all the solutions to the equation in an interval of length $2\pi$ (or 360). In Figure 1.29, these solutions are marked with $x$s and correspond to the graph intersections marked with solid dots.

Having done that, we then recall that $\sin(x) = \sin(x + 2\pi)$ (or $\sin(x) = \sin(x + 360)$). That is, $x$ is a solution if and only if $x + 2\pi$ (or $x + 360$) is a solution. So, step two, having found two solutions we then make more by adding and subtracting multiples of $2\pi$ (or 360).

Let’s demonstrate how this works with the equation

$$2\sin(x) = 1.$$ 

First we solve the equation using what you may regard as your usual method. We divide both sides by 2.

$$\sin(x) = 1/2.$$ 

Then we think, hmmm, what value of $x$ makes that true? Oh, $x = \pi/6$ and $x = 5\pi/6$ (or $x = 30$ and $x = 150$). To accomplish step two we add multiples of $2\pi$ (or 360).
\[ x = \pi/6 + k2\pi \text{ for any integer } k \text{ and } \]
\[ x = 5\pi/6 + k2\pi \text{ for any integer } k \]

Thus we can fill in the values on the graph as follows. In degrees the solution looks like the following.

\[ x = 30 + k360 \text{ for any integer } k \text{ and } \]
\[ x = 150 + k360 \text{ for any integer } k \]

Here is another example.

**Find all the solutions to the following** \( 2\cos(x) = \sqrt{3} \).

**Answer:** first we divide both sides by 2 to get

\[ \cos(x) = \frac{\sqrt{3}}{2}. \]
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Now we recall $x = \pi/6$ and $x = -\pi/6$ both solve this equation (or $x = 30$ and $x = -30$ degrees). Then we add multiples of $2\pi$ to get the following solution.

\[
x = \pi/6 + k2\pi \text{ for any integer } k \text{ and }
\]

\[
x = -\pi/6 + k2\pi \text{ for any integer } k
\]

or in degrees

\[
x = 30 + k360 \text{ for any integer } k \text{ and }
\]

\[
x = -30 + k360 \text{ for any integer } k.
\]

Here is yet another example with another added layer of manipulation.

Find all the solutions to the following $2\sin(3x) = \sqrt{3}$.

Answer: first we divide both sides by 2 to get

\[
\sin(3x) = \frac{\sqrt{3}}{2}.
\]

Then, to help us focus, we let $z = 3x$ and write

\[
\sin(z) = \frac{\sqrt{3}}{2}.
\]

Now we recall $z = \pi/3$ and $z = 2\pi/3$ both solve this equation (or $z = 60$ and $z = 120$ degrees). Then we add multiples of $2\pi$ to get the following solution.

\[
z = \pi/3 + k2\pi \text{ for any integer } k \text{ and }
\]

\[
z = 2\pi/3 + k2\pi \text{ for any integer } k
\]

Now we recall $z = 3x$, so

\[
3x = \pi/3 + k2\pi \text{ for any integer } k \text{ and }
\]
3x = 2\pi/3 + k2\pi for any integer k

and dividing by 3 we get

\[ x = \pi/9 + k2\pi/3 \] for any integer k and
\[ x = 2\pi/9 + k2\pi/3 \] for any integer k.

In degrees

\[ z = 60 + k360 \] for any integer k and
\[ z = 120 + k360 \] for any integer k

\[ z = 3x, \text{ so} \]
\[ 3x = 60 + k360 \] for any integer k and
\[ 3x = 120 + k360 \] for any integer k

or

\[ x = 20 + k120 \] for any integer k and
\[ x = 40 + k120 \] for any integer k

**Problems.** Find all the solutions to the following.

1. \( \sin(x) = -1 \)
2. \( \cos(x) = 0 \)
3. \( \tan(x) = 1 \)
4. \( 2\tan(x) = \sqrt{3} \)
5. \( 4\cos(x) = 2 \)
6. \( \sin(x - 10^o) = 1/2 \)
7. \( 4\sqrt{3}\cos(\pi x) = 6 \)
8. \( \sin(2x + \pi/7) = -1 \) (in radians)
9. \( \cos(2x - 15^o) = 0 \)
10. \( \tan(5x) = 1 \)
11. \( 2\tan(x/2) = \sqrt{3} \)
12. \( 4\cos(x/4) = 2 \)
13. \( \sin(3x - 1) = 1/2 \) (in radians)
1.5 Identities and the Return to Differentiation

1.5.1 Polynomial Identities

Polynomial identities are just algebraic rearrangements of each other. For example,

\[ 3x^2 + x = x(3x + 1). \]

When we write this, we mean if you take any value for \( x \), say \( x = 3 \), and you plug that value into the left side and you plug it into the right side, you get the same result on both sides. And this works for every possible real number, \( x \).

Identities should not be confused with equations. The following is an equation.

\[ x^2 - 2x + 1 = x^2 - 1. \]

Why is it an equation, but not an identity? Well, take \( x = 3 \). Plug it into the left,

\[ 3^2 - 2 \cdot 3 + 1, \]

and you get 4. Plug \( x = 3 \) into the right and you get

\[ 3^2 - 1 \]

or 8. It is an equation so you can solve it to find values of \( x \) which do make the equation true, but not every real number makes the equation true. (In fact, the equation is false for most numbers.) For example, we have just seen that \( x = 3 \) does not make it true. You can check that cancelling the \( x^2 \)'s and solving for \( x \), you see that \( x = 1 \) makes this equation true, but that is the only value of \( x \) that does.

In summary, \textit{not every equation is an identity}. (In fact, most equations are not identities.)
Our goal here is to be able to show which equations are identities and which are not.

1. To show an equation is not an identity, one must find a value of \( x \) for which the equation is false. (To elaborate a bit, \( x \) must make sense in both sides and those senses (those values) should disagree, that is they should be different numbers.)

2. To show an equation is an identity, one must use algebra to show the two sides of the equation are equal.

Example 34 Show that \( x^2 + 3x + 1 = x^4 + x \) is not an identity.

Solution. Pick a number, say \( x = 1 \) since it is easy to calculate (generally, if one number will work, lots will work). On the left we have \( 1^2 + 3 \cdot 1 + 1 = 5 \). On the right we have \( 1^4 + 1 = 2 \). \( 5 \neq 2 \), so this is not an identity.

Example 35 Show that \((x - 2)(x + 3) = x^2 + x - 6\) is an identity.

Solution. We must use algebra to show the left is equal to the right. We multiply (FOILing to some).

\[
(x - 2)(x + 3) = x \cdot x + 3x - 2x - 2 \cdot 3 = x^2 + 3x - 2x - 6 = x^2 + x - 6
\]

There, we see the left is equal to the right.

Problems: For the following determine whether the following are identities, or not. (If not, then you must find a number for which the two sides are not equal. If so, then you must use algebra to show they are the same.)

Problem 36 Determine whether \((x + 1)^2 = x^2 + 2x + 1\) is an identity.

Problem 37 Determine whether \((x - 1)(x + 1) = x^2 - 1\) is an identity.
1.5. IDENTITIES AND THE RETURN TO DIFFERENTIATION

Problem 38  Determine whether \((x - 4)(x + 3) = x^2 + 6\) is an identity.

Problem 39  Determine whether \(x(2x^2 + 1) = 2x^3 + x\) is an identity.

Problem 40  Determine whether \((x^2 - 1)(x^2 + 1) = x^4 - 1\) is an identity.

Problem 41  Determine whether \((x^2 - 2x + 1)(x + 1) = x^3 + x + 1\) is an identity.

1.5.2 Symmetry Identities

The easiest trigonometric identities to verify are probably the symmetry identities. The following are examples of such.

\[
\sin(-\theta) = -\sin(\theta) \quad \text{(1.15)}
\]
\[
\cos(-\theta) = \cos(\theta) \quad \text{(1.16)}
\]

Again, we use the circle definition of sine and cosine. Let’s look at that circle. We see \(\sin(\theta)\) and \(\sin(-\theta)\) highlighted on the picture (see Figure 1.31). The angle \(-\theta\) is just the reflection across the \(x\)-axis of \(\theta\), and \(\sin(-\theta)\) is just the reflection of \(\sin(\theta)\) across the \(x\)-axis also. Thus, \(\sin(-\theta)\) is just the negative of \(\sin(\theta)\). With the same picture and a similar analysis we see the cosine identity, \(\cos(-\theta) = \cos(\theta)\). These are called reflection identities because the symmetry involves a reflection across the \(x\)-axis. We also might note that the above identities show us that sine is an odd function \((-f(x) = f(x))\) and cosine is an even function \((f(x) = f(-x))\).

Other examples of symmetry identities are the following.

\[
\sin(\theta) = -\sin(\theta + 180^\circ) \quad \text{or} \quad \sin(\theta) = -\sin(\theta + \pi) \quad \text{(1.17)}
\]
\[
\cos(\theta) = -\cos(\theta + 180^\circ) \quad \text{or} \quad \cos(\theta) = -\cos(\theta + \pi) \quad \text{(1.18)}
\]
These can be analyzed by examining the effect of a rotation of 180° (or \(\pi\)) on the picture used to define sine and cosine. What we notice is that, the angle, the ray, and the line segments involved in the definition of \(\sin(\theta)\) are all above the \(x\)-axis, while the angle, the ray, and the line segments involved in the definition of \(\sin(\theta + 180)\) are a 180° rotation of the corresponding elements from the first part. Thus, where the line segment corresponding to \(\sin(\theta)\) points up, the line segment that corresponds to \(\sin(\theta + 180)\) points down. Consequently, \(\sin(\theta)\) and \(\sin(\theta + 180)\) have the same absolute value, but one is the negative of the other. Thus, \(\sin(\theta) = -\sin(\theta + 180°)\).

The argument for the cosines is similar. These are known as rotation identities.

**Example 42** Determine whether \(\tan(\theta) = \tan(\theta + 180°)\) is an identity.

**Solution.** We will show this is an identity using the rotation symmetries of sine and cosine. To do so, then we must start with the LHS (\(\tan(\theta)\)) and show by a sequence
of algebraic manipulations that we end up with the RHS (tan(θ + 180°)). So, let’s do this.

\[
\text{(LHS) } \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{-\sin(\theta + 180)}{-\cos(\theta + 180)} \quad \text{using the identity to substitute} \quad (1.20)
\]

\[
= \frac{+\sin(\theta + 180)}{+\cos(\theta + 180)} \quad \text{multiplying top and bottom by } -1 \quad (1.21)
\]

\[
= \tan(\theta + 180) \quad \text{(RHS)} \quad (1.22)
\]

Thus our equation is an identity.

It seems like there ought to be a negative in that identity. Last year, in class, the popular vote was for the alternative equation, tan(θ) = −tan(θ + 180°) because it seemed like there should be a negative in that identity. But, we have shown that, nooo, there is no negative.

**Example 43** Determine whether tan(θ) = −tan(θ + 180°) is an identity.

**Solution.** We know better now. We will show that this is not an identity. To do so we will use the number x = 45° (or x = π/4) (Arron’s choice). On the left tan(45°) = 1. On the right −tan(45° + 180°) = −tan(225°) = −1 So, the LHS and the RHS are not equal. Conclusion, our equation is not an identity.

**Problem 44** Determine whether tan(θ) = tan(−θ) is an identity.

**Problem 45** Determine whether tan(θ) = −tan(−θ) is an identity.

We have one more set of reflection identities. These are formed by reflection across the y-axis, and thus are reflection identities. Their validity may be established much
as the first set of symmetry identities were.

\[
\sin(180^\circ - \theta) = \sin(\theta) \quad \text{or} \quad \sin(\pi - \theta) = \sin(\theta) \quad (1.23)
\]

\[
\cos(180^\circ - \theta) = -\cos(\theta) \quad \text{or} \quad \cos(\pi - \theta) = -\cos(\theta) \quad (1.24)
\]

We have one more set of rotation identities. These are formed by a rotation of 90°.

\[
\sin(\theta + 90^\circ) = \cos(\theta) \quad \text{or} \quad \sin(\theta + \pi/2) = \cos(\theta) \quad (1.25)
\]

\[
\cos(\theta + 90^\circ) = -\sin(\theta) \quad \text{or} \quad \cos(\theta + \pi/2) = -\sin(\theta) \quad (1.26)
\]

### 1.5.3 Circle Identities

As in the polynomial case, equations which involve the trigonometric functions fall into two categories, those equations which are true for all values, and those which are only true for some values. For example, the following trigonometric equation is \textit{not} an identity.

\[
\sin(x) = 0 \quad (1.27)
\]

Why? Well, the only values of \( x \) which make it true are the integer multiples of \( \pi \). That is, \( x = k\pi \) for \( k = 0, \pm 1, \pm 2, \ldots \) This is in contrast with the next equation, Equation 1.28.

In class we derived the following identity from the definition of sine and cosine (using the circle).

\[
\sin^2(x) + \cos^2(x) = 1 \quad (1.28)
\]

This equation is true for all values of \( x \). For example, take \( x = 30^\circ \). \( \sin(30) = 1/2 \),
and $\cos(30) = \sqrt{3}/2$.

\[
\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} + \frac{3}{4} = 1.
\]

A similar calculation will work for any value of $x$. One can convince oneself of this by examining the circular picture used to define sine and cosine (as we did in class). It is because Equation 1.28 is true for all $x$ that we say Equation 1.28 is an identity.

Dividing Equation 1.28 by $\cos^2(x)$ we get another identity.

\[
\tan^2(x) + 1 = \sec^2(x)
\] (1.29)

Dividing Equation 1.28 by $\sin^2(x)$ we get yet another identity.

\[
1 + \cot^2(x) = \csc^2(x)
\] (1.30)

**Example 46** Determine whether $\cos(x) \tan(x) = \sin(x)$ is an identity.

**Solution.** In class it was observed that $\tan(x) = \frac{\sin(x)}{\cos(x)}$. This allowed us to do the following.

\[
\begin{align*}
\text{(LHS) } \cos(x) \tan(x) &= \cos(x) \frac{\sin(x)}{\cos(x)}, \text{ cancelling the } \cos(x)s \\
&= \sin(x) \quad \text{(RHS)}
\end{align*}
\] (1.31)

Starting with the LHS, and proceeding algebraically, we obtain the RHS. This is what we have to do to show we have an identity. We’ve done it, so we have an identity. ■

**Example 47** Determine whether $\sin^2(x) - \cos^2(x) = -1$ is an identity.

**Solution.** After some discussion, we determined in class that plugging $60^\circ$ in resulted in the following. Because $\sin(60) = \sqrt{3}/2$ and $\cos(60) = 1/2$ our equation yields

\[
\sin^2(60) - \cos^2(60) = \left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^2 = \frac{3}{4} - \frac{1}{4} = \frac{1}{2} \neq -1
\]
Our equation is false for \( x = 60^\circ \), thus the equation is not an identity; it is merely an equation.

**Problem 48** Determine whether \( \sin^2(x)(1 + \cot^2(x)) = 1 \) is an identity.

**Problem 49** Determine whether \( \frac{1 - \sin^2(x)}{\cos(x)} = \cos(x) \) is an identity.

**Problem 50** Determine whether \( \cos^2(x)(\tan^2(x) - 1) = 1 \) is an identity.

**Problem 51** Determine whether \( \frac{1 - \sin^2(x)}{\cos(x)} = \cos(x) \) is an identity.

**Problem 52** Determine whether \( \cot(x) \sin(x) = \cos(x) \) is an identity.

**Problem 53** Determine whether \( \cot(x) \csc(x) = \cos(x) \) is an identity.

### 1.5.4 Addition Identities

For us, the purpose for the identities discussed in this section will be found in the next section, when we discuss the derivative of the sine function and cosine function. Given the time, one could explore many additional applications. In class we will establish the following identities.

\[
\sin(x + y) = \sin(x) \cos(y) + \sin(y) \cos(x) \tag{1.33}
\]
\[
\cos(x + y) = \cos(y) \cos(x) - \sin(x) \sin(y) \tag{1.34}
\]
\[
\tan(x + y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x) \tan(y)} \tag{1.35}
\]

**Example 54** Using the subtraction identity that we just proved, show that \( \cos(x + y) = \cos(y) \cos(x) - \sin(x) \sin(y) \) is an identity.
Solution We use $y = -y'$ in Equation 1.45 to write the second line.

\[
\begin{align*}
\text{(LHS)} \quad \cos(x + y') &= \cos(x - (-y')) \\
&= \cos(-y') \cos(x) + \sin(x) \sin(-y') \\
&= \cos(y') \cos(x) + (-\sin(x) \sin(y')) \\
&= \cos(y') \cos(x) - \sin(x) \sin(y') \quad \text{(RHS)}
\end{align*}
\]

where we use $\cos(y') = \cos(-y')$ and $-\sin(y') = \sin(-y')$ in to get the third line. Because $y'$ was arbitrary, we can just call it $y$, and we are done.

The other addition identities can be derived from these cosine addition identities. A limited utility of the addition identities is the ability to derive exact the values of the cosine of some more angles.

**Example 55** Find the exact value of $\cos(15^\circ)$.

**Solution.** Notice $15 = 45 - 30$. So $\cos(15^\circ) = \cos(45 - 30)$. By Equation 1.45 we have.

\[
\cos(15) = \cos(45 - 30) = \cos(45) \cos(30) + \sin(45) \sin(30) \quad \text{(1.40)}
\]

\[
= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{\sqrt{3} + 1}{2\sqrt{2}} \quad \text{(1.41)}
\]

The so-called double angle identities can be obtained from the addition identities.

**Example 56** Show $\sin(2x) = 2 \sin(x) \cos(x)$ is an identity.

**Solution**

\[
\begin{align*}
\text{(LHS)} \quad \sin(2x) &= \sin(x + x) \\
&= \sin(x) \cos(x) + \sin(x) \cos(x) \\
&= 2 \sin(x) \cos(x) \quad \text{(RHS)}
\end{align*}
\]
Problem 57 Show \(\cos(2x) = \cos^2(x) - \sin^2(x)\) is an identity.

Problem 58 Show \(\cos(2x) = 2\cos^2(x) - 1\) is an identity.

Problem 59 Show \(\cos(2x) = 1 - 2\sin^2(x)\) is an identity.

1.5.5 The addition identity: why is it what it is?

There will be two aspects to our discussion. 1. Why do these identities have the form they have, and 2, how do we use these identities to verify new identities.

Part 1. Let us examine the cosine addition identity. Why does it have the form it does? Every identity has a path of reasoning that leads to it, and some paths are easier than others. For the addition identities, the easiest path begins by showing

\[
\cos(x - y) = \cos(y)\cos(x) + \sin(x)\sin(y)
\]  

(1.45)

and this is done most easily by assuming (in degrees) \(0 \leq y \leq x \leq 180\) and using the definition associated with the unit circle, see Figure 1.33. Assuming the circle in Figure 1.33 is a unit circle, we can calculate the distance between the points \(A\) and

![Figure 1.33: \(\cos(x - y) = \cos(x)\cos(y) + \sin(x)\sin(y)\)](image)
1.5. IDENTITIES AND THE RETURN TO DIFFERENTIATION

$B$ in two different ways; the above identity comes out of setting the two expression for the distance equal to each other. Let us see.

First, focus on the part of Figure 1.33 that only has $A$ and $B$ in it, see Figure 1.34. For part 1, we can calculate the distance between $A$ and $B$ by observing that

Figure 1.34: $A$ and $B$, part one

the horizontal distance between $A$ and $B$ is $\cos(x) - \cos(y)$ and the vertical distance between $A$ and $B$ is $\sin(x) - \sin(y)$. Using Pythagorus’s theorem, we can calculate the square of the distance $d^2$ between $A$ and $B$ as

$$d^2 = (\cos(x) - \cos(y))^2 + (\sin(x) - \sin(y))^2 \quad (1.46)$$
$$= \cos^2(x) - 2\cos(x)\cos(y) + \cos^2(y) + \sin^2(x) - 2\sin(x)\sin(y) + \sin^2(y) \quad (1.47)$$
$$= \cos^2(x) + \sin^2(x) + \cos^2(y) + \sin^2(y) - 2\cos(x)\cos(y) - 2\sin(x)\sin(y) \quad (1.48)$$
$$= 1 + 1 - 2\cos(x)\cos(y) - 2\sin(x)\sin(y) \quad (1.49)$$

On the other hand, for part 2, we can focus on the triangle formed by the origin and the points $A$ and $B$, and apply the law of cosines, see Figure 1.35. The law of cosines says

$$d^2 = 1^2 + 1^2 - 2 \cdot 1 \cdot 1 \cos(x - y) \quad (1.50)$$
$$= 2 - 2\cos(x - y) \quad (1.51)$$
Setting the two parts equal to each other (since they both equal $d$) we get

$$1 + 1 - 2 \cos(x) \cos(x) - 2 \sin(x) \sin(y) = 2 - 2 \cos(x - y)$$

Simplifying we have

$$-2(\cos(x) \cos(x) + \sin(x) \sin(y)) = -2 \cos(x - y)$$

Cancelling the $-2$s we get the desired identity, Equation 1.45.

1.5.6 Identities Continued

There are lots of identities, in fact they go on forever. Just for completeness, here are a couple further, more common identities. You should be aware of their existence. (All good citizens of the empire are aware of their existence. As I am sure you are aware, they are the foundation of the traditional tea ceremony where-in the number of individuals partaking in the ceremony must be distributed around the circular ceremony table in a uniform manner with...)

1.5.7 Double angle identities

$$\sin(2x) = 2 \sin(x) \cos(x)$$  \hspace{1cm} (1.52)
\[ \cos(2x) = \cos^2(x) - \sin^2(x) \quad (1.53) \]
\[ \cos(2x) = 2\cos^2(x) - 1 \quad (1.54) \]
\[ \cos(2x) = 1 - 2\sin^2(x) \quad (1.55) \]

### 1.5.8 Half angle identities

\[
\begin{align*}
\cos(x) &= \pm \sqrt{\frac{\cos(x) + 1}{2}} \quad \text{with the sign determined by the angle} \\
\sin(x) &= \pm \sqrt{\frac{1 - \cos(x)}{2}} \quad \text{with the sign determined by the angle}
\end{align*}
\quad (1.56) \quad (1.57)
\]

**Example 60**  Show \( \sin\left(\frac{x}{2}\right) = \pm \sqrt{\frac{1 - \cos(x)}{2}} \) is an identity, where the appropriate sign (±) is determined by the quadrant that \( \frac{x}{2} \) lies in.

**Solution.** To do this we start with the identity \( \cos(2y) = 1 - 2\sin^2(y) \) and let \( y = \frac{x}{2} \), so \( 2y = x \). Ready? Here we go.

\[
\cos(x) = \cos(2y) = 1 - 2\sin^2(y) = 1 - 2\sin^2\left(\frac{x}{2}\right) \quad (1.58)
\]

Now we rearrange to get

\[ 2\sin^2\left(\frac{x}{2}\right) = 1 - \cos(x) \]

Dividing by 2 and taking the square root we get

\[
\sin\left(\frac{x}{2}\right) = \pm \sqrt{\frac{1 - \cos(x)}{2}}.
\]

Both the + and − values of the square root will be consistent with the previous equation, and only one will be correct, so we need to check which sign is the correct one for the angle \( \frac{x}{2} \).
Problem 61  Show \( \cos\left(\frac{x}{2}\right) = \pm \sqrt{\frac{\cos(x) + 1}{2}} \) is an identity, where the appropriate sign \((\pm)\) is determined by the quadrant that \( \frac{x}{2} \) lies in.

Example 62  Calculate the exact value of \( \cos(7.5^\circ) \).

Solution. Before we begin, let us just note that because both 7.5\(^\circ\) and 15\(^\circ\) are in the first quadrant, that \( \cos(7.5^\circ) \) and \( \cos(15^\circ) \) are both positive. Next, we apply the half angle formula twice. The first time we get

\[
\sin(15^\circ) = \sqrt{\frac{\cos(30) + 1}{2}}
\]

\[
= \sqrt{\frac{\sqrt{3} + 1}{2}}
\]

\[
= \sqrt{\frac{\sqrt{3} + 2}{4}} = \sqrt{\frac{\sqrt{3} + 2}{2}}
\]

We repeat the process

\[
\sin(15^\circ) = \sqrt{\frac{\cos(15) + 1}{2}}
\]

\[
= \sqrt{\frac{\left(\frac{\sqrt{3} + 2}{2}\right) + 1}{2}}
\]

\[
= \sqrt{\frac{\left(\frac{\sqrt{3} + 2}{4}\right) + \frac{2}{4}}{4}}
\]

\[
= \sqrt{\frac{\sqrt{3} + 2 + 2}{2}}
\]

You might wonder, why would we do this? Our calculator gives a pretty good value for \( \cos(15^\circ) \), no? It’s true, though it is not exact. Sometimes one needs the exact
value of a number. Exactly how many angels do dance on the head of that pin? No, not that one, the other one, the yellow pin. For the moment, an excellent purpose for this exercise is to practice the process. Now, do the next problems.

Problem 63 Calculate the exact value of $\sin(7.5^\circ)$.

Problem 64 Calculate the exact value of $\tan(7.5^\circ)$.  

1.6 Differentiation

1.6.1 The Derivative of Sine

We are now ready to compute the derivative of sine. So, what are we doing? Recall, the derivative tells us about the slope of a function. At the moment we are discussing $f(x) = \sin(x)$. So, $f'(x)$ will tell us the rate at which $f(x)$ goes up or goes down. For example, we know $f(\frac{\pi}{4}) = \frac{d}{dx} \sin(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$. $f'(\frac{\pi}{4}) = \sin(\frac{\pi}{4})$ will tell us at what rate the graph of $\sin(x)$ is going up (or down) at the value of $x = \frac{\pi}{4}$. If we draw a line tangent to the graph of $\sin(x)$ at the point $(\frac{\pi}{4}, \frac{1}{\sqrt{2}})$, then $f'(\frac{\pi}{4})$ is the slope of that tangent line. See Figure 1.36.

Figure 1.36: The Derivative of $\sin(x)$.

Let’s skip to the chase. Here is a summary of the basic trigonometric differentiation formulas.

\[
\begin{align*}
\frac{d}{dx} \sin(x) &= \cos(x) & \frac{d}{dx} \csc(x) &= -\cot(x) \csc(x) \\
\frac{d}{dx} \cos(x) &= -\sin(x) & \frac{d}{dx} \sec(x) &= \tan(x) \sec(x) \\
\frac{d}{dx} \tan(x) &= \sec^2(x) & \frac{d}{dx} \cot(x) &= -\csc^2(x)
\end{align*}
\] 

(1.59) (1.60) (1.61)
Naturally, in order to acquire proper civil development, the good citizen must experience the derivation of at least one of these derivatives. Your experience will take place shortly, in the next subsection or two. For the moment let us apply these to find the equations of lines tangent to the graph of sine and the graph of cosine.

**Example 65** Find the equation of the line tangent to \( \sin(x) \) at \( x = \pi/4 \).

**Solution.** Recall, last semester we could find the equation of a tangent line at \( a \) using the following

\[
l(x) = f(a) + f'(a)(x - a).
\]  

(1.62)

We do the same here. \( a = \pi/4, \ f(x) = \sin(x), \) and \( f'(x) = \cos(x) \). So \( f(a) = \sin(\pi/4) = 1/\sqrt{2} \), and \( f'(a) = \cos(\pi/4) = 1/\sqrt{2} \). So,

\[
l(x) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}(x - \frac{\pi}{4}).
\]

Just for entertainments sake, let’s graph these two functions. Yes, indeed, that looks like the desired tangent line.

**Example 66** Use the quotient rule of differentiation to derive the derivative of \( \tan(x) \) from the derivatives of \( \sin(x) \) and \( \cos(x) \).
Solution. Recall, $\tan(x) = \frac{\sin(x)}{\cos(x)}$. Also, recall the quotient rule.

$$
\left( \frac{f(x)}{g(x)} \right)' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}
$$

So,

$$
\left( \frac{\sin(x)}{\cos(x)} \right)' = \frac{\cos(x) \cos(x) - \sin(x)(-\sin(x))}{(\cos(x))^2} = \frac{\cos^2(x) + \sin(x)\sin(x)}{\cos^2(x)} = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x)
$$

(1.63) (1.64) (1.65) (1.66) (1.67)

Problem 67 Find the equation of the line tangent to $\sin(x)$ at $x = \pi/3$.

Problem 68 Find the equation of the line tangent to $\cos(x)$ at $x = \pi/4$.

Problem 69 Find the equation of the line tangent to $\tan(x)$ at $x = 2\pi/3$.

Problem 70 Find the equation of the line tangent to $\sin(x)$ at $x = \pi/6$.

Problem 71 Find the equation of the line tangent to $\cos(x)$ at $x = 5\pi/6$.

Problem 72 Find the equation of the line tangent to $\tan(x)$ at $x = \pi/4$.

Problem 73 Find the equation of the line tangent to $\cot(x)$ at $x = 3\pi/4$.

Problem 74 Find the equation of the line tangent to $\sec(x)$ at $x = 5\pi/6$.

Problem 75 Find the equation of the line tangent to $\csc(x)$ at $x = \pi/3$. 
1.6. DIFFERENTIATION

1.6.2 The Generalized Trigonometric Rules of Differentiation

There are also generalized trigonometric rules of differentiation.

\[
\frac{d}{dx} \sin(g(x)) = \cos(g(x))g'(x) \tag{1.68}
\]

\[
\frac{d}{dx} \cos(g(x)) = -\sin(g(x))g'(x) \tag{1.69}
\]

\[
\frac{d}{dx} \tan(g(x)) = \sec^2(g(x))g'(x) \tag{1.70}
\]

\[
\frac{d}{dx} \sec(g(x)) = \sec(g(x)) \tan(g(x))g'(x) \tag{1.71}
\]

\[
\frac{d}{dx} \csc(g(x)) = -\csc(g(x)) \cot(g(x))g'(x) \tag{1.72}
\]

\[
\frac{d}{dx} \cot(g(x)) = -\csc^2(g(x))g'(x) \tag{1.73}
\]

These follow from the chain rule. In class a couple of these should be derived from the chain rule.

Example 76 Find the derivative of \( f(x) = \sin(x^3 + x) \).

Solution. The function \( g(x) = x^3 + x \), so \( g'(x) = 3x^2 + 1 \). Following the generalized trigonometric rule for sine we get \( f'(x) = \cos(x^3 + x) \cdot (3x^2 + 1) \).

Example 77 Find the equation of the tangent line to \( f(x) = \sin(\pi \sqrt{x+1}) \) at \( x = 0 \).

Solution. The function \( g(x) = \pi \sqrt{x+1} \), so \( g'(x) = \frac{\pi}{2\sqrt{x+1}} \). Following the generalized trigonometric rule for sine we get \( f'(x) = \cos(\pi \sqrt{x+1}) \cdot \frac{\pi}{2\sqrt{x+1}} \). The next step is to evaluate \( f(0) \) and \( f'(0) \). We get \( f(x) = 0 \) and \( f'(0) = \cos(\pi \sqrt{0+1}) \cdot \frac{\pi}{2\sqrt{0+1}} = (-1)\frac{\pi}{2} = -\pi/2 \).

Problem 78 Find the derivatives of the following.
Problem 79  Find the equation of the line tangent to \( \sin(x^2) \) at \( x = \sqrt{\pi/3} \).

Problem 80  Find the equation of the line tangent to \( \sin(\pi x) \) at \( x = 1/4 \).

Problem 81  Find the equation of the line tangent to \( \cos(\pi x/180) \) at \( x = 30 \).

Problem 82  Find the equation of the line tangent to \( \tan(\sqrt{x}) \) at \( x = \pi^2/4 \).

1.6.3 Derivative of Sine: Why is it what it is?

The definition of the derivative is the limit of a difference quotient. Recall, in Calculus with Polynomials, in order to calculate the derivative of, say \( f(x) = x^2 \), we setup a difference quotient. (Recall this represented the slope of the secant line that passed through the graph of \( f(x) = x^2 \) at the points that corresponded to the values of \( x \) and \( x + h \).)

\[
\frac{f(x + h) - f(x)}{h} = \frac{(x + h)^2 - x^2}{h} \tag{1.74}
\]

If we took the limit as \( h \to 0 \) of this, the first thing we would have tried would have been to take the limit of the top expression and the limit of the bottom expression,
1.6. DIFFERENTIATION

separately. If we did so we would only get \( \frac{0}{0} \) which doesn’t do us any good. So we employed some algebra, to get an expression that had a limit we could make sense of. Recall,

\[
\frac{(x + h)^2 - x^2}{h} = \frac{x^2 - 2xh + h^2 - x^2}{h} = \frac{2xh + h^2}{h} = \frac{h(2x + h)}{h} = 2x + h \tag{1.75}
\]

This we can take a limit of.

We will do the same thing here. Here our difference quotient is the following.

\[
\frac{f(x + h) - f(x)}{h} = \frac{\sin(x + h) - \sin(x)}{h} \tag{1.76}
\]

If we just take the limit as \( h \to 0 \) of the top expression and the bottom expression separately we again get \( \frac{0}{0} \), which doesn’t do us any more good now than it did before.

In order to make sense of this difference quotient we will employ algebra in the form of the addition identity for sine. But first, we need a couple preliminary calculations.

Recall that earlier we showed that \( \lim_{h \to 0} \frac{\sin(h)}{h} = 1 \). We would like to show a similar limit holds for cosine.

A preliminary exercise. Claim:

\[
\lim_{h \to 0} \frac{\cos(h) - 1}{h} = 0
\]

To show this, consider

\[
\frac{\cos(h) - 1}{h} = \frac{\cos(h) - 1}{h} \cdot \frac{\cos(h) + 1}{\cos(h) + 1} = \frac{(\cos(h) - 1)(\cos(h) + 1)}{h(\cos(h) + 1)} = \frac{(\cos^2(h) - 1)}{h(\cos(h) + 1)} = \frac{-\sin^2(h)}{h(\cos(h) + 1)} = \frac{-\sin(h) \cdot \sin(h)}{h \cos(h) + 1}
\]
Taking the limits we get (using \( \lim_{h \to 0} \frac{\sin(h)}{h} = 1 \))

\[
\lim_{h \to 0} \frac{\cos(h) - 1}{h} = \lim_{h \to 0} \frac{-\sin(h)}{h} \cdot \lim_{h \to 0} \frac{\sin(h)}{\cos(h) + 1} = -\lim_{h \to 0} \frac{\sin(h)}{h} \cdot \lim_{h \to 0} \frac{\sin(h)}{\lim_{h \to 0} \cos(h) + 1} = -1 \cdot \frac{0}{1 + 1} = 0
\]

We are now ready to calculate the derivative of \( \sin(x) \). Let \( f(x) = \sin(x) \). From the definition of the derivative we have that

\[
f'(x) = \lim_{h \to 0} \frac{\sin(x + h) - \sin(x)}{h}.
\]

For \( x = 0 \) we have

\[
f'(x) = \lim_{h \to 0} \frac{\sin(h) - \sin(0)}{h} = \lim_{h \to 0} \frac{\sin(h)}{h}.
\]

Earlier, by considering the circular definition of sine we showed this limit was 1. That is \( \frac{d}{dx} \sin(0) = 1 \)

Because of the addition identity for sine we have

\[
\sin(x + h) = \sin(x) \cos(h) + \sin(h) \cos(x)
\]

Plugging this into the derivative definition we get

\[
\lim_{h \to 0} \frac{\sin(x + h) - \sin(x)}{h} = \lim_{h \to 0} \frac{\sin(x) \cos(h) + \sin(h) \cos(x) - \sin(x)}{h} = \lim_{h \to 0} \frac{\sin(x) \cos(h) - \sin(x) + \sin(h) \cos(x)}{h}
\]
\[\begin{align*}
\text{1.6. DIFFERENTIATION} \\
&= \lim_{h \to 0} \frac{\sin(x)(\cos(h) - 1)}{h} + \lim_{h \to 0} \frac{\sin(h) \cos(x)}{h} \\
&= \lim_{h \to 0} \frac{\sin(x) \cos(h) - \sin(x)}{h} + \lim_{h \to 0} \frac{\sin(h) \cos(x)}{h} \\
&= \sin(x) \lim_{h \to 0} \frac{\cos(h) - 1}{h} + \cos(x) \lim_{h \to 0} \frac{\sin(h)}{h} \\
&= \sin(x) \cdot 0 + \cos(x) \cdot 1 \\
&= \cos(x)
\end{align*}\]

The derivative of \(\cos(x)\) can be obtained in an analogous manner.
1.7 Antidifferentiation and Integration

Our previous area calculations and antidifferentiation did not involve trigonometric functions, because we had not discussed them yet. But now we have.

Example 83 Find the antiderivative of \( f(x) = \sin(x) \).

Solution: Well, we need \( F(x) \) so that \( F'(x) = \sin(x) \). That is almost \( \cos(x) \), but cosine’s derivative is \(-\sin(x)\). So, we need \( F(x) = -\cos(x) \). We check. \( F'(x) = -(-\sin(x)) = \sin(x) \). OK

\[
F(x) = -\cos(x) + C.
\]

Knowing how to find the antiderivatives of trigonometric functions requires knowing the derivatives of the trigonometric functions backwards and forwards.

Example 84 Find the antiderivative of \( f(x) = \sec^2(x) \).

Solution: Recall the derivative of \( \tan(x) \) is \( \sec^2(x) \). So, let \( F(x) = \tan(x) + C \).

Example 85 Find the area under the graph of the function \( f(x) = \sin(x) \) between the values of \( x = 0 \) and \( x = \pi \).

Solution: We recall the area under the curve is given by \( \int_0^\pi \sin(x) \, dx \), and we recall that \( \int_a^b f(x) \, dx = F(b) - F(a) \) where \( F'(x) = f(x) \), that is, \( F(x) \) is the antiderivative
of $f(x)$. Because $f(x) = \sin(x)$, it follows that $F(x) = -\cos(x)$. Why?, well check that $F'(x) = -\sin(x) = \sin(x)$. So,

$$
\int_0^\pi \sin(x) \, dx = F(\pi) - F(0) = -\cos(\pi) - (-\cos(0)) = -(1) - (1) = 2.
$$

□

Example 86 Find the area under the graph of the function $f(x) = \cos(x)$ between the values of $x = \pi/4$ and $x = \pi/3$.

Solution: The area under the curve is given by $\int_{\pi/4}^{\pi/3} \cos(x) \, dx$, and $F(x)$, the antiderivative of $f(x) = \cos(x)$, is $F(x) = \sin(x)$. So,

$$
\int_{\pi/4}^{\pi/3} \cos(x) \, dx = F(\pi/3) - F(\pi/4) = \sin(\pi/3) - \sin(\pi/4) = \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} = \frac{1}{2}(\sqrt{3} - \sqrt{2}).
$$

□
CHAPTER 1. THE THREE PICTURES OF TRIGONOMETRY

Antidifferentiation Problems.

1. Find the area under the graph of $f(x) = \sin(x)$ between the values of $x = \pi/4$ and $x = \pi/2$.

2. Find the area under the graph of $f(x) = \sin(x)$ between the values of $x = 0$ and $x = \pi/3$.

3. Find the area under the graph of $f(x) = \sin(x)$ between the values of $x = -\pi/4$ and $x = \pi/4$.

4. Find the area under the graph of $f(x) = \sin(x)$ between the values of $x = 0$ and $x = \pi/4$.

5. Find the area under the graph of $f(x) = \cos(x)$ between the values of $x = -\pi/4$ and $x = \pi/4$.

6. Find the area under the graph of $f(x) = \cos(x)$ between the values of $x = -\pi/3$ and $x = \pi/2$. 
1.8 What the successful student will be able to do

1. Understand how to measure angles
2. Be able to convert angles expressed one set units to another
3. Understand the circle based definition of sine and cosine
4. Be able to demonstrate that understanding by finding the sine and cosine when given necessary information
5. Know the sine and cosine of the standard special angles, $0^\circ, 30^\circ, 45^\circ, 60^\circ, 90^\circ$, and their multiples
6. Know what the graph of sine (and cosine) looks like
7. Know the meaning of midline, amplitude, and frequency
8. Know how to raise, lower, shift up, shift down, change the amplitude and change the period of the graph of sine
9. Know how to solve right triangle problems
10. Know the law of sines
11. Know how to use the law of sines to solve triangle problems, and be aware of complications that can arise
12. Know the law of cosines
13. Know how to use the law to solve triangle problems
14. Know the difference between and identity and an equation
15. Know (in principle) how to show an equation is an identity or not
16. Know the fundamental trigonometric identities
   (a) the circular (Pythagorean) identities
   (b) the symmetry identities
   (c) the rotational identities
   (d) the addition identities
17. Know how to use identities to show equations are identities
18. Know the derivatives of $\sin(x)$ and $\cos(x)$.
19. Know how calculate the derivatives of $\tan(x)$, $\csc(x)$, $\sec(x)$, and $\cot(x)$ from the derivatives of $\sin(x)$ and $\cos(x)$. 
20. have some sense of why \( \lim_{h \to 0} \frac{\sin(h)}{h} = 1 \)

21. be able to find the equation of a line tangent to \( \sin(x) \) or \( \cos(x) \)

22. know the generalized differentiation rules for trigonometric functions

23. be able to find the equations of tangent lines of functions involving trigonometric functions

24. be able to linearize functions involving trigonometric functions

25. be able to find antiderivatives of some trigonometric functions

26. be able to find the area under the graphs of some trig functions