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Chapter 4

Exponential Functions I

4.1 The Function $2^x$.

We now consider exponential functions. We are familiar of the powers of 2.

$2, 2^2, 2^3, 2^4$

We recall that to write $2^n$ we mean to convey the multiplication together of $n$ twos. We frequently indicate this multiplication by writing the 2s down in a row. For example,

$2^7 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$.

It should be fairly clear that if we write $m$ 2s down in a row, and then follow this with $n$ 2s in a row that there will be a total of $m + n$ 2s in a row. For example

$2^3 \cdot 2^5 = (2 \cdot 2 \cdot 2) \cdot (2 \cdot 2 \cdot 2 \cdot 2 \cdot 2) = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^{3+5}$.

Because this works for every non-negative pair of integers, $m$ and $n$, we have a formula

$2^m \cdot 2^n = 2^{m+n}$. (4.1)

By a non-negative integer, we mean $m \geq 0$ and $n \geq 0$.

We didn’t discuss this in class, but what if $m$ is negative? That is, what if $m = -n$? Recall we write $2^{-n}$ to mean $\frac{1}{2^n}$. Then

$2^k \cdot 2^{-n} = 2^k \cdot \frac{1}{2^n} = \frac{2^k}{2^n}$.
We have a fraction with $k$ 2s on top and $n$ 2s on the bottom, and there will be a bunch of cancellation.

For example,

$$2^3 \cdot 2^{-7} = 2^3 \cdot \frac{1}{2^7} = \frac{2 \cdot 2 \cdot 2}{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2} = \frac{1}{2 \cdot 2 \cdot 2}.$$  

Or

$$2^7 \cdot 2^{-3} = 2^7 \cdot \frac{1}{2^3} = \frac{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}{2 \cdot 2} = 2 \cdot 2 \cdot 2.$$  

In general, if $k \geq n$ we will have $k - n$ 2s on top, and if $n \geq k$ we will have $n - k$ 2s on the bottom. We can write both outcomes as $2^{k-n}$.

As a rule this can be written as follows

$$2^k \cdot 2^{-n} = 2^{k-n}.$$  

We began by letting $m = -n$, and if we use this to rewrite $k - n = k + m$ and if we substitute these expressions into the above equation, we get

$$2^k \cdot 2^m = 2^{k+m}$$  

which looks just like Equation 4.1 except that $m$ could now be either positive, negative or zero. In a similar fashion we can show Equation 4.1 holds whenever $m$ and $n$ are both either positive, negative, or zero, that is, whenever $m$ and $n$ are any integer.

In fact, as we discussed in class, Equation 4.1

$$2^r \cdot 2^s = 2^{r+s}$$  

holds for any rational numbers $r$ and $s$, though it takes a little more effort to show this. In fact, it holds for any numbers $\alpha$ and $\beta$

$$2^\alpha \cdot 2^\beta = 2^{\alpha+\beta}.$$  

We didn’t talk about why this is so in class, but it is because we can approximate any number with a rational number, and the equation holds with those approximating rational numbers. So, not only does it make sense to talk about

$$2^{3/4} \text{ and } 2^{1/2}$$
4.2. $F(X) = A^X$ for $A > 1$

it makes sense to talk about

$2^\pi$ and $2^{\pi/2}$

The graph of $f(x) = 2^x$ looks like the following. How do we know that? Well, we graph

![Figure 4.1: The Graph of $f(x) = 2^x$.](image)

the values of $f(x) = 2^x$ in much the same process as we analyzed them above. We start by plotting the values of $x$ that are integers. Then we graph values of $x$ that are rational (fractions). After that we recall that $f(x) = 2^x$ is increasing, so the graph is always going up as we move right.

4.2 $f(x) = a^x$ for $a > 1$

In fact, this line of reasoning can be pushed even farther. For any positive number, or zero, call it $a$, and any two numbers $\alpha$ and $\beta$, not only does

$$a^\alpha$$

make sense, if we accept the convention $a^{-\alpha} = \frac{1}{a^\alpha}$, then we have the law of exponents

$$a^\alpha \cdot a^\beta = a^{\alpha + \beta}.$$

An artifact of the convention is that $a^0 = 1$. Why is this an artifact? Because

$$1 = a^\alpha \cdot \frac{1}{a^\alpha} = a^\alpha \cdot a^{-\alpha} = a^{\alpha - \alpha} = a^0.$$

Like any construction, one can start from many places. Some will argue that continuity implies $a^0 = 1$ and that with the law of exponents, the convention follows.

Frequently, these results are placed in a convenient summary box, and called the laws of exponents.
THE LAWS of EXPONENTS

For \( a \geq 0 \), and \( \alpha, \beta \in \mathbb{R} \)

\[
\begin{align*}
    a^0 &= 1, \\
    a^{-\alpha} &= \frac{1}{a^\alpha} \\
    a^\alpha \cdot a^\beta &= a^{\alpha+\beta} \\
    (a^\alpha)^\beta &= a^{\alpha \cdot \beta}
\end{align*}
\]

As special cases which technically are consequences of the above, but occur frequently enough to merit note, we have the following.

Special cases

\[
\begin{align*}
    \frac{a^\alpha}{a^\beta} &= a^{\alpha-\beta} \\
    \sqrt[n]{a^s} &= a^{s/r} \text{ where } r > 0
\end{align*}
\]

Problems Do the problems from the handout.

4.3 The graphs of exponential functions.

For \( a > 1 \) the graph of \( f(x) = a^x \) looks like the following.

Figure 4.2: The Graph of \( f(x) = a^x \) for \( a > 1 \).

Notice there are two important points, \((0,1)\) and \((1,a)\). They are important for two reasons. First, they help identify the function, from the value of \( a \). Second, if you are graphing the function by hand, they help you draw the graph.

Do we expect to see this graph increasing? Compare the values of \( f(x) \) and \( f(x+1) \).
4.4. \( F(X) = A^X \) FOR \( A > 1 \)

Which should be bigger? Because the graph is increasing, the graph suggests that \( f(x + 1) \) should be bigger than \( f(x) \). Do we expect that from the function?

If \( a > 1 \), then \( b \cdot a > b \) for any \( b > 0 \). For \( a > 1 \), consider

\[
f(x + 1) = a^{x+1} = a^x \cdot a^1 > a^x = f(x).
\]

So, yes, we expect the graph to be increasing.

To do a complete analysis we need to compare \( f(x) \) and \( f(x + h) \) for any positive \( h \).

Recall a graph is *increasing* if and only if for every \( x < y \) we have \( f(x) < f(y) \). This can be restated as follows (let \( h = y - x \)). A graph is *increasing* if and only if for every \( x \) and for every \( h > 0 \), \( f(x) < f(x + h) \). For any \( x \) and \( h > 0 \), we consider

\[
f(x + h) = a^{x+h} = a^x \cdot a^h \text{ compared to } a^x = f(x).
\]

If we know that \( a^h > 1 \) for all \( h > 0 \) we will have our answer (for \( a > 1 \)). I will let you ponder why this is the case. (It is true but it takes some time and consideration to show why.)

### 4.4 \( f(x) = a^x \) for \( a > 1 \)

For \( 0 < a < 1 \) the graph of \( f(x) = a^x \) looks like the following. Again, we notice there are two important points, \((0,1)\) and \((1,a)\). Because \( a < 1 \) we expect to see this graph decrease.

![Figure 4.3: The Graph of \( f(x) = a^x \) for \( 0 < a < 1 \).](image)
CHAPTER 4. EXPONENTIAL FUNCTIONS I

4.4.1 Problems

1. Sketch the graph of \( f(x) = 3^x \).
2. Sketch the graph of \( f(x) = 1.2^x \).
3. Sketch the graph of \( f(x) = 5^x \).
4. Sketch the graph of \( f(x) = (\sqrt{2})^x \).
5. Sketch the graph of \( f(x) = \left(\frac{1}{2}\right)^x \).
6. Sketch the graph of \( f(x) = \left(\frac{1}{3}\right)^x \).
7. Sketch the graph of \( f(x) = \left(\frac{2}{5}\right)^x \).
8. Sketch the graph of \( f(x) = .2^x \).
9. What does the graph of \( f(x) = a^x \) look like for \( a = 1 \)?

4.5 New exponential graphs from old exponential graphs

Recall, the graph of \( g(x) = f(x) + k \) is the same as the graph of \( f(x) \) shifted up \( k \) units (down if \( k \) is negative). The graph of \( g(x) = f(x - k) \) is the same as the graph of \( f(x) \) shifted right \( k \) units (left if \( k \) is negative). So \( g(x) = 2^x + 1 \) looks the same as the graph of \( 2^x \) shifted up by 1, and \( h(x) = 2^{x-1} \) looks like the graph of \( 2^x \) shifted right by 1 unit.

On the other hand \( g(x) = f(x)/2 \) looks like the graph of \( f(x) \) where the vertical distance is shrunk by a factor of 2, so the graph of \( k(x) = 2^x/2 \) looks like the graph of \( 2^x \) with the vertical height shrunk by a factor of 2. Because of the law of exponents \( 2^{x-1} = 2^x/2 \) so, shifting the graph of \( 2^x \) right by 1 unit is the same as shrinking the vertical height by a factor of 2. For exponential functions, moving the graph left and right is like rescaling the vertical axis (stretching or shrinking the graph).
4.6 \( e^x \) and Differentiation

We recall that to calculate the derivative of a function we compute the following.

\[
\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
\]

If the limit exists, then that is what we mean by \( f'(x) \). For example, if \( f(x) = 2^x \) we compute

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad (4.2)
\]

\[
= \lim_{h \to 0} \frac{2^{x+h} - 2^x}{h} \quad (4.3)
\]

\[
= \lim_{h \to 0} \frac{2^x 2^h - 2^x}{h} \quad (4.4)
\]

\[
= \lim_{h \to 0} \frac{2^x (2^h - 1)}{h} \quad (4.5)
\]

\[
= 2^x \lim_{h \to 0} \frac{2^h - 1}{h} \quad (4.6)
\]

But what is

\[
\lim_{h \to 0} \frac{2^h - 1}{h}?
\]

That is a little tricky to say.

In class we addressed that by looking at the graph of

\[
\frac{2^x - 1}{x},
\]

Figure 4.4: The Graph of \( f(x) = \frac{2^x - 1}{x} \) near \( x = 0 \).
CHAPTER 4. EXPONENTIAL FUNCTIONS I

shown in Figure 4.4. It is not clear what value we see at $x = 0$, but whatever it is, that’s what we want.

In class we approached this by asking another question. We asked, for what value of $a$ would the limit be 1? That is,

$$\lim_{h \to 0} \frac{a^h - 1}{h} = 1?$$

Whatever $a$ is here we see it should be bigger than 2. In class we tried different values and got pretty close with 2.7182.

We call this number $e$, whatever its value is. (We see above $e \approx 2.7182$.) It has been proven that $e$ is not a rational number. In fact, it has been proven that $e$ is not the root of a polynomial! See Section 4.8 for another approach to finding $e$.

Okay,..., you may say, soooo why do we care?

Ah!, right. Well, consider $f(x) = e^x$. This function has a really nice derivative. Check this out.

We calculate

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$  \hspace{2cm} (4.8)

$$= \lim_{h \to 0} \frac{e^{x+h} - e^x}{h}$$  \hspace{2cm} (4.9)

$$= \lim_{h \to 0} \frac{e^x e^h - e^x}{h}$$  \hspace{2cm} (4.10)

$$= \lim_{h \to 0} \frac{e^x(e^h - 1)}{h}$$  \hspace{2cm} (4.11)

$$= e^x \lim_{h \to 0} \frac{e^h - 1}{h}$$  \hspace{2cm} (4.12)

$$= e^x \cdot 1 \quad \text{because that’s the way $e$ is,} \quad \lim_{h \to 0} \frac{e^h - 1}{h} = 1$$  \hspace{2cm} (4.13)

$$= e^x.$$  \hspace{2cm} (4.14)

The derivative of $e^x$ is $e^x$. The function and its derivative are the same. How nice is that?

**Example 1** Find the equation of the line tangent to $f(x) = 3 \cdot e^x + 2x$ at $x = 0$. 
Solution. We need a slope and a point. To find the point we need \((0, f(0))\).

\[ f(0) = 3 \cdot e^0 + 2 \cdot 0 = 3 \cdot 1 + 0 = 3. \]

To find the slope we need \(m = f'(0)\). Using the derivative of \(e^x\), we have

\[ f'(x) = 3 \cdot e^x + 2. \]

Plugging in \(x = 0\) we get \(f'(0) = 3 \cdot e^0 + 2 = 3 + 2 = 5\). For the equation of the tangent line we put this together to get

\[ y = 3 + (5)(x - 0) = 3 + 5x. \]

If \(e^x\) is its own derivative, then it is also its own antiderivative.

\[ \int e^x \, dx = e^x + C \]

Example 2  Find the area underneath the graph of \(f(x) = 3 \cdot e^x + 1\) between the values of \(x = 0\) and \(x = 2\).

Solution. We want to find \(A^2_0(3 \cdot e^x + 1) = \int_0^2 3 \cdot e^x + 1 \, dx\). First we find an antiderivative of \(f(x)\). \(F(x) = 3e^x + x\). Now

\[ \int_0^2 3 \cdot e^x + 1 \, dx = F(2) - F(0) = 3e^2 + 2 - (3e^0 + 0) = 3e^2 - 1 \approx 21.17. \]
4.7 Generalized Exponential Differentiation.

For \( f(x) = e^x \), the derivative \( f'(x) = e^x \). There is also a generalized differentiation rule for \( e^x \) which goes like this.

For \( f(x) = e^{g(x)} \), the derivative \( f'(x) = e^{g(x)} g'(x) \).

Example 3 Find the derivative of \( f(x) = e^{x^3+2x} \).

Solution. Following the generalized differentiation rule with \( g(x) = x^3 + 2x \) (and hence \( g'(x) = 3x^2 + 2 \)), we have \( f'(x) = e^{x^3+2x}(3x^2 + 2) \).

Example 4 Find the derivative of \( f(x) = e^{\sin(x)} \).

Solution. Following the generalized differentiation rule with \( g(x) = \sin(x) \) (and hence \( g'(x) = \cos(x) \)), we have \( f'(x) = e^{\sin(x)} \cos(x) \).

Example 5 Find the equation of the line tangent to \( f(x) = 2e^{x^2-x} + x \) at \( x = 1 \).

Solution. We need a slope and a point. To find the point we need \((1, f(1))\).

\[
 f(1) = 2e^{1^2-1} + 1 = 2e^0 + 1 = 2 \cdot 1 + 1 = 3. 
\]

To find the slope we need \( m = f'(1) \). Using the generalized exponential rule, we have

\[
 f'(x) = 2e^{x^2-x}(2x - 1) + 1. 
\]

Plugging in \( x = 1 \) we get \( f'(1) = 2e^{1^2-1}(2 \cdot 1 - 1) + 1 = 2 \cdot 1(1) + 1 = 3 \). Putting this together we have

\[
 y = 3 + 3(x - 1). 
\]

for the equation of the tangent line.
4.7. GENERALIZED EXPONENTIAL DIFFERENTIATION.

4.7.1 Problems.

Problem 6  Compute the derivatives of the following.

a. \( f(x) = e^{5x} \)  
   i. \( h(x) = e^{\sin x} \)  

b. \( f(x) = 3e^{5x} \)  
   j. \( h(x) = e^{\cos x} \)

b. \( f(x) = 7e^{5x} + 2x \)  
   k. \( f(x) = e^{\sin(x+1)} \)

c. \( f(x) = e^{5x+7} \)  
   l. \( f(x) = e^{\tan(x)} \)

d. \( f(x) = e^{7x+5} \)  
   m. \( f(x) = e^{x\sin(x)} \)

e. \( f(x) = e^{x^2+5} \)  
   n. \( f(x) = \frac{e^x - e^{-x}}{2} \)

f. \( f(x) = e^{2x^3} \)  
   o. \( f(x) = \frac{e^x + e^{-x}}{2} \)

g. \( f(x) = e^{3x+x^2} \)
   p. \( f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} \)

Problem 7  Find the equation of the line tangent to the graph of \( f(x) = e^{5x} \) above the value \( a = 0 \).

Problem 8  Find the equation of the line tangent to the graph of \( f(x) = e^x \) above the value \( a = 1/2 \).

Problem 9  Find the equation of the line tangent to the graph of \( f(x) = e^{x^2+5} \) above the value \( a = 0 \).

Problem 10  Find the equation of the line tangent to the graph of \( f(x) = e^{x^2-5} \) above the value \( a = 1 \).

Problem 11  Find the equation of the line tangent to the graph of \( f(x) = e^{x^2-5} \) above the value \( a = 2 \).

Problem 12  Find the area underneath the graph of \( f(x) = e^x \) between the values of \( x = 1 \) and \( x = 3 \).

Problem 13  Find the area underneath the graph of \( f(x) = 2e^x - 1 \) between the values of \( x = 0 \) and \( x = 1 \).

Problem 14  Find the area underneath the graph of \( f(x) = 1 - e^x \) between the values of \( x = 0 \) and \( x = 1 \).
4.8 The Value of e and a Perfect Slope ($m = 1$)

We now attempt to convince ourselves, by trial and error that for the value of $a \sim 2.781$ that the graph of $a^x$ has a slope of 1 at $x = 0$. We start out this exploration by examining the graphs of $f(x) = 2^x$, $g(x) = 4^x$, and of $y = x + 1$ (which has a slope equal to 1).

Here is what we saw. The slope of $f(x) = 2^x$ at $x = 0$ is too small, and the slope of $g(x) = 4^x$ is too large. So, we choose a number between 2 and 4, and try this again. After some trial and error we end up looking at $f(x) = 2.7^x$, and $g(x) = 2.8^x$. Hmmm, well it’s a little hard to see which is largest, never mind which, if either, is larger than $y = x + 1$. Lets try zooming in a little bit. Hmm, that didn’t help.

We could try subtracting $y = x + 1$ from all three equations. What would that do you say? Let’s see. Hmm, much better. So, what does this tell us? Figure 4.8 tells us that for...
4.8. THE VALUE OF E AND A PERFECT SLOPE ($M = 1$)

$x > 0$, $f(x) < x + 1$ and $g(x) > x + 1$, because $f(x) > -(x + 1) < 0$ and $g(x) - (x + 1) > 0$. That is, 2.8 is too big for a value of $a$ and 2.7 is too small for a value of $a$.

Now wait a second, you may say. How did you know to pick values of $a$ between 2.7 and 2.8. The reason is that these were the first values where one ($a = 2.7$) was too small and the other ($a = 2.8$) was too large. If we picked values between 2 and 4 (in increments of 0.1) all the values between 2 and 2.7 would be too small, with $a^x - (x + 1) < 0$ for $x > 0$. All the values between 2.8 and 4 would be too big, with $a^x - (x + 1) > 0$ for $x > 0$. For example, in the following figure we see the graphs of $a^x - (x + 1)$ for values of $a$ between 2.5 and 2.8. The
values for $a = 2.5$ and $a = 2.6$ are definitely too small, and the value of $a = 2.8$ is too large, but $a = 2.7$ is too close to tell. In Figure 4.8, where the scale is a little more appropriate, it is clearer that the value of $a = 2.7$ is too small. If one has doubts about this we could zoom in a little closer, as in Figure 4.10, to remove all doubt.

Figure 4.10: Zooming in a little closer.

We could continue on, for example, moving to the values of $a = 2.71$ and $a = 2.72$, and then on to $a = 2.7181$ and $a = 2.7182$. In the first case we see we need something larger than 2.71 and smaller than 2.72. In the second case we see we need something larger than 2.7181 and smaller than 2.7182. It is not so easy to show but there is a number $a$ for which

Figure 4.11: Continuing the process.

\[ a^x - (x + 1) \]

neither goes above the $x$-axis, nor below the $x$-axis, at $x = 0$.

That would mean $a^x - (x + 1) \geq 0$ for all $x$, with equality when $x = 0$. But that would mean $y = x + 1$ is tangent to $a^x$, because $a^x \geq x + 1$ for all $x$ with equality for $x = 0$. (This might not be completely clear, but trust me, it works, even if the justification is a little
terse.) Thus, the derivative of $a^x$ at $x = 0$ equals the derivative of $y = x$ at $x = 0$, which is 1. Thus we have for our special value of $a$,

$$\text{for } f(x) = a^x \text{ we have } f'(0) = 1 \quad (4.15)$$

Traditionally, the special value of $a$ is given the name $e$, and roughly $e \approx 2.718281828459$. In fact, in second semester Calculus (Caalc II) you will learn that

$$e = 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{n!} + \cdots$$

where $n! = n \cdot (n - 1) \cdots 3 \cdot 2 \cdot 1$.

For $f(x) = e^x$, when we say that $f'(0) = 1$, the definition of the derivative implies that we are saying the number $e$ has the property that

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1 \quad (4.16)$$

The number $e$ is the only number with that property.
Chapter 5

Inverse Functions and Logarithms

5.1 Inverse Functions: What they are, in general.

Let us begin with a completely incomprehensible rendition. In retrospect, it will be perfectly clear, but the first time through it may be somewhat challenging. We say the function $h(x)$ is the inverse of the function $f(x)$ if

$$h(f(x)) = x \text{ and } f(h(x)) = x. \quad (5.1)$$

This is sometimes put in the following equivalent way. We say $h(x)$ and $f(x)$ are inverses if for each $a$ (in the domain of $h$) we have

$$h(a) = b \text{ and } f(b) = a. \quad (5.2)$$

(The presence of $a$ in domain of $h$ is equivalent to $b$ being in the domain $f$.) It would be best to see some friendly examples.

5.2 Algebraic Examples of Inverse Functions

Example 15 Check that $f(x) = \frac{1}{x-1}$ and $h(x) = \frac{x+1}{x}$ are inverses.
To check that they are inverses we must do as Equation 5.1 tells us, and show $f(h(x)) = x$ and $h(f(x)) = x$. We’ll do the first, you can do the second.

$$f(h(x)) = \frac{1}{h(x) - 1}$$

$$= \frac{1}{\frac{x+1}{x} - 1}$$

$$= \frac{1}{\frac{x+1}{x} - \frac{x}{x}}$$

$$= \frac{1}{\frac{x+1-x}{x}}$$

since $1 = \frac{x}{x}$

$$= \frac{1}{\frac{x+1-x}{x}}$$

using their common denominator

$$= \frac{1}{1} = 1 \cdot \frac{x}{1} = x$$

where the last line uses the invert and multiply rule for the division of fractions,

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \cdot \frac{d}{c} \quad (5.8)$$

Now, why do we call $f(x)$ and $h(x)$ inverses? Take $2 = 1$ and plug this into $f(x)$ to get $f(2) = \frac{1}{2-1} = 1$. Then plug that (1) into $h(x)$ to get $h(1) = \frac{1+1}{1} = \frac{2}{1} = 2$. Notice

$$f(1) = 2 \text{ and } h(2) = 1$$

The function $h$ undoes what $f$ does, and vise-versa. This is why $h$ and $f$ are called inverses.

**Problem 16** For $f(x) = \frac{1}{x-1}$ and $h(x) = \frac{x+1}{x}$ show $h(f(x)) = x$, completing the demonstration that $f(x)$ and $h(x)$ are inverses. (Notice, this is the reverse of Example 1, $f(x)$ is the inside function and $h(x)$ is the outside function.)

**Problem 17** Check that $f(x) = \frac{2x+1}{3x}$ and $h(x) = \frac{1}{3x-2}$ are inverses.

Frequently, to show two functions are inverses we will be lazy and just show one of $f(h(x)) = x$ or $h(f(x)) = x$. Frequently, if one is true, so is the other. But not always.
Example 18  Like Rudolf, the Red Nosed Reindeer (the most famous reindeer of all), our most famous inverse function of all is,...? Yes, let \( f(x) = x^2 \) for \( x \geq 0 \) and let \( h(x) = \sqrt{x} \) (also with \( x \geq 0 \)).

Examining the compositions, we see they behave as expected

\[
 f(h(x)) = (\sqrt{x})^2 = x
\]

and

\[
 h(f(x)) = \sqrt{x^2} = x.
\]

Notice that the second composition works because of the restriction of the domain of \( f(x) \) to those numbers \( x \geq 0 \). Without the restriction, the first equation is true but the second is not.

How does one know what the inverse of a function is, or whether it even exists? A first step in that direction is the following.

5.2.1 Finding Algebraic Inverses.

Example 19 Find the inverse of \( f(x) = \frac{x + 1}{x - 1} \).

To do this, our goal is to find a \( y \) (that depends upon \( x \), so we could write \( y = y(x) \)) such that \( f(y) = \frac{y + 1}{y - 1} = x \). The function \( y(x) \) is our function \( h(x) \). In other words, we have

\[
 \frac{y + 1}{y - 1} = x
\]

and we want to solve for \( y \). OK, multiply by \( y + 1 \) to get

\[
 y + 1 = x(y - 1).
\]

Keep multiplying

\[
 y + 1 = x(y - 1) = xy - x.
\]

Put the \( ys \) on the left, and the rest on the right

\[
 y - xy = -x - 1.
\]
Factor out the $y$,
\[ y(1 - x) = -x - 1, \]
and divide, to get our final answer
\[ y = \frac{-x - 1}{1 - x} = \frac{-(x + 1)}{1 - x} = \frac{x + 1}{x - 1}. \]
We see that $y$ depends upon $x$. We write
\[ h(x) = \frac{x + 1}{x - 1}. \]

Well, that’s a little bizarre, $f(x) = h(x)$ is its own inverse. However, it is not completely bizarre. We know other functions that are their own inverses. Don’t we?

**Problem 20** Check that $f(x) = \frac{x + 1}{x - 1}$ and $h(x) = \frac{x + 1}{x - 1}$ are inverses.

**Problem 21** Find the inverse of $f(x) = \frac{x + 2}{x + 3}$.

**Problem 22** Find the inverse of $f(x) = \frac{2x - 1}{2x + 1}$.

Finally, a somewhat abstract example.

**Example 23** Find the inverse of $f(x) = \frac{ax + b}{cx + d}$. (There are some technical details: $x \neq -d/c$; we will need $ad - bc \neq 0$.)

To do this, our goal is to find a $y$ (that depends upon $x$, so we could write $y = y(x)$) such that $f(y) = \frac{ay + b}{cy + d} = x$. The function $y(x)$ is our function $h(x)$. In other words, we have
\[ ay + b = x(cy + d) \tag{5.10} \]
and we want to solve for $y$. OK, multiply by $cy + d$ to get
\[ ay + b = x(cy + d). \]
Keep multiplying
\[ ay + b = x(cy + d) = cxy + xd. \]
Put the $y$s on the left, and the rest on the right

$$ay - cxy = xd - b.$$ 

Factor out the $y$,

$$y(a - cx) = xd - b,$$

and divide, to get our final answer

$$y = \frac{xd - b}{a - cx}.$$ 

We see that $y$ depends upon $x$. We write

$$h(x) = \frac{xd - b}{a - cx}.$$ 

**Problem 24** Check that $f(x) = \frac{ax + b}{cx + d}$ and $h(x) = \frac{xd - b}{a - cx}$ are inverses. (We assume $x \neq -\frac{d}{c}$, nor $x \neq \frac{a}{c}$ and that $ad - bc \neq 0$.)

### 5.3 The Graphs of Inverse Functions

The graph of a function $f(x)$ and the graph of its inverse, say $h(x)$, have a nice relationship. Can you see it?

![Graph of Function and Inverse](image)

There are three relevant facts needed to convince oneself, and perhaps to convince someone else, of the relationship between the graph of a function and the graph of its inverse.

1. The graph of a function $h(x)$ contains all pairs $(a, b)$ where $b = h(a)$. 

---

**Figure 5.1**: The Graphs of $f(x) = \frac{1}{x - 1}$ and Its Inverse $h(x) = \frac{x + 1}{x}$. 

---
2. For \( f(x) \) and \( h(x) \) inverses we have \( h(a) = b \iff f(b) = a \).

3. In the cartesian coordinate system the points \( P = (a, b) \) and \( Q = (b, a) \) are reflections of each other across the line \( y = x \) (provided the scales of the \( x \)-axis and the \( y \)-axis are the same).

What means this term reflection across the line \( y = x \)? In the figure below the blue face is constructed from the red by reflecting the red face across the line \( y = x \). The purpose of the black line segment connecting the mouths is to demonstrate that whenever corresponding parts are connected with such a line segment, that line segment is perpendicular to (and bisected by) the line \( y = x \).

Figure 5.2: A reflection across the line \( y = x \).

In class we convinced ourselves of the following.

**Lemma 25** A function \( h(x) \) is the inverse of the function \( f(x) \) if and only if the graph of \( h(x) \) is the reflection across the line \( y = x \) of the graph of \( f(x) \).
5.3. THE GRAPHS OF INVERSE FUNCTIONS

Figure 5.3: The Graphs of a Function and Its Inverse.

5.3.1 When can a Function have an Inverse?

In class we decided that a function failed to have an inverse when it failed to be *one to one*, that is, when it failed the *horizontal line test*. Our reasoning went as follows. We can always take the graph of \( f(x) \) and reflect it across the line \( y = x \). That image will be the graph of a function when and only when it doesn’t fail the *vertical line test*. That reflection will fail the vertical line test precisely when the original graph fails the horizontal line test. There you go. We gave a name to the function whose graph we constructed by reflecting \( f(x) \)’s graph. We called it the inverse of \( f(x) \) and frequently called the inverse \( h(x) \), or sometimes \( f^{-1}(x) \).

We acknowledge that \( f^{-1}(x) \) is a *misleading* notation because it looks like

\[
f^{-1}(x) = \frac{1}{f(x)} \text{ but this is wrong!}
\]
5.4 The Natural Logarithm Function

Having an inverse function is very useful. It helps us solve equations. For example, solve

\[ f(x) = 3. \]

If we had the inverse function \( h(x) \) lying around, this would be easy to do. First we take the inverse of both sides.

\[ h(f(x)) = h(3) \]

Being inverses \( h(f(x)) = x \), so we have

\[ x = h(3). \]

In short, all we have to do is plug the number 3 into the inverse \( h \).

5.4.1 The definition and the graph

If we recall the graph of \( f(x) = e^x \) (in Figure 5.4), one thing we will notice is that it satisfies the horizontal line test. So, it has an inverse, which we will denote \( h(x) = \log_e(x) \). The graph of \( \log_e(x) \) is in Figure 5.5. The graph of \( \log_e(x) \) is the reflection of the graph of \( e^x \) across the line \( y = x \) because the two functions are inverses.
5.4. THE NATURAL LOGARITHM FUNCTION

Figure 5.5: The Graph of \( f(x) = \log_e(x) \).

5.4.2 The Derivative

For \( h(x) = \ln(x) \), we have \( h'(x) = \frac{1}{x} \).

The generalized logarithmic differentiation rule is,

for \( h(x) = \ln(g(x)) \), the derivative \( h'(x) = \frac{g'(x)}{g(x)} \).

Example 26  Differentiate \( h(x) = \ln(x^2 + 1) \).

Solution. By the generalized logarithmic differentiation rule, \( h'(x) = \frac{2x}{x^2 + 1} \).

Example 27  Find the equation of the line tangent to the function \( h(x) = \ln(x^2 + 1) \) above the value \( a = 1 \).

Solution. We need a point and a slope. The point is \((1, h(1))\). What do we use for \( h(1) \)?

Well, plugging 1 in we get \( h(1) = \ln(1^2 + 1) = \ln(2) \sim 0.6931 \). To be most accurate, we will use \( \ln(2) \).

The slope is \( m = h'(1) \). We calculated the derivative in the problem above, so we have \( h'(1) = \frac{2 \cdot 1}{1^2 + 1} = \frac{2}{2} = 1 \).

Together, we have \( y - \ln(2) = 1 \cdot (x - 1) \), or \( y = \ln(2) + x - 1 \).

Problem 28  Compute the derivatives of the following.
a. \( f(x) = \ln(x^2 + 1) \)  

b. \( f(x) = \ln(x^3 + 1) \)  
c. \( f(x) = \ln(x^2 + x) \)  
d. \( f(x) = \frac{\ln(x)}{x^2 + 1} \)

Problem 29
Find the equation of the line tangent to the graph of \( f(x) = \ln(x^2 + x) \) above the value \( a = 1 \)

Problem 30
Find the equation of the line tangent to the graph of \( f(x) = \ln(2x^2 - 1) \) above the value \( a = 1 \)

5.5 The Inverse Differentiation Rule.

From the relation between the graphs of inverse functions we get the following.

Theorem 31 (The Inverse Derivative Rule) For \( f \) and \( h \) inverses, and \( a \) and \( b \) such that \( h(a) = b \) and \( f(b) = a \) we have

\[
h'(a) = \frac{1}{f'(b)}.
\]

In the graph below we can see this makes sense.

Figure 5.6: The Graphs of a Function and Its Inverse.
Chapter 6

Exponents and Logarithms base $b$

6.1 The Special Case $a = 2$.

For $f(x) = 2^x$, the derivative $f'(x) = 2^x \ln(2)$.

There is also a generalized differentiation rule for $2^x$ which is very similar to the case $a = e$.

For $f(x) = 2^{g(x)}$, the derivative $f'(x) = 2^{g(x)} \cdot \ln(2)g'(x)$.

Example 32 Find the derivative of $f(x) = 2^{x^3+2x}$.

Solution. Following the generalized differentiation rule with $g(x) = x^3 + 2x$ (and hence $g'(x) = 3x^2 + 2$), we have $f'(x) = 2^{x^3+2x} \cdot \ln(2)(3x^2 + 2)$.

Example 33 Find the derivative of $f(x) = 2\sin(x)$.

Solution. Following the generalized differentiation rule with $g(x) = \sin(x)$ (and hence $g'(x) = \cos(x)$), we have $f'(x) = 2\sin(x) \cdot \ln(2)\cos(x)$.

Example 34 Find the equation of the line tangent to $f(x) = 3 \cdot 2^{x^2-x} + x$ at $x = 1$.

Solution. We need a slope and a point. To find the point we need $(1, f(1))$.

$$f(1) = 3 \cdot 2^{1^2-1} + 1 = 3 \cdot 2^0 + 1 = 3 \cdot 1 + 1 = 4.$$
To find the slope we need \( m = f'(1) \). Using the generalized exponential rule, we have

\[
f'(x) = 3 \cdot 2^{x^2-x} \cdot \ln(2)(2x - 1) + 1.
\]

Plugging in \( x = 1 \) we get

\[
f'(1) = 3 \cdot 2^{1^2-1} \cdot \ln(2)(2 \cdot 1 - 1) + 1 = 3 \cdot 2^0 \cdot \ln(2)(1) + 1 = 3 \ln(2) + 1.
\]

For the equation of the tangent line we put this together to get

\[
y = 3 + (3 \ln(2) + 1)(x - 1).
\]

**Problem 35** Compute the derivatives of the following.

a. \( f(x) = 2^{5x} \)

b. \( f(x) = 7 \cdot 2^{5x} \)

c. \( f(x) = 5 \cdot 2^{5x} + 3x \)

d. \( f(x) = 2^{5x+7} \)

e. \( f(x) = 2^{7x+5} \)

f. \( f(x) = 2^{x^2+5} \)

g. \( f(x) = 2^{2x^3} \)

h. \( f(x) = 2^{3x+x^2} \)

i. \( h(x) = 2^{\sin x} \)

j. \( h(x) = 2^{\cos x} \)

k. \( f(x) = 2^{\sin(x+1)} \)

l. \( f(x) = 2^{\tan(x)} \)

m. \( f(x) = 2^{x \sin(x)} \)

n. \( f(x) = \frac{2^x - 2^{-x}}{2} \)

o. \( f(x) = \frac{2^x + 2^{-x}}{2} \)

p. \( f(x) = \frac{2^x - 2^{-x}}{2^x + 2^{-x}} \)

**Problem 36**

Find the equation of the line tangent to the graph of \( f(x) = 2^{5x} \) above the value \( a = 0 \)

**Problem 37**

Find the equation of the line tangent to the graph of \( f(x) = 2^{x^2+5} \) above the value \( a = 0 \)

**Problem 38**

Find the equation of the line tangent to the graph of \( f(x) = 2^{x^2-5} \) above the value \( a = 1 \)
6.1. THE SPECIAL CASE $A = 2$.

Problem 39
Find the equation of the line tangent to the graph of $f(x) = 2^{x^2 - 5}$ above the value $a = 2$

6.1.1 The General Case $a > 0$.

As we have seen, for $f(x) = a^x$, the derivative $f'(x) = a^x \ln(a)$.

The generalized differentiation rule for $f(x) = a^{g(x)}$, is the derivative

$$f'(x) = a^{g(x)} \cdot \ln(a)g'(x).$$

Example 40 Find the derivative of $f(x) = 5^{x^3 + 2x}$.

Solution. Following the generalized differentiation rule with $g(x) = x^3 + 2x$ (and hence $g'(x) = 3x^2 + 2$), we have $f'(x) = 5^{x^3 + 2x} \cdot \ln(5)(3x^2 + 2)$.

Example 41 Find the derivative of $f(x) = 11^{\sin(x)}$.

Solution. Following the generalized differentiation rule with $g(x) = \sin(x)$ (and hence $g'(x) = \cos(x)$), we have $f'(x) = 11^{\sin(x)} \cdot \ln(11) \cos(x)$.

Example 42 Find the equation of the line tangent to $f(x) = 6 \cdot 7^{x^2 - x} + x$ at $x = 1$.

Solution. We need a slope and a point. To find the point we need $(1, f(1))$.

$$f(1) = 6 \cdot 7^{1^2 - 1} + 1 = 6 \cdot 7^0 + 1 = 6 \cdot 1 + 1 = 7.$$ To find the slope we need $m = f'(1)$. Using the generalized exponential rule, we have

$$f'(x) = 6 \cdot 7^{x^2 - x} \cdot \ln(7)(2x - 1) + 1.$$ Plugging in $x = 1$ we get $f'(1) = 6 \cdot 7^{1^2 - 1} \cdot \ln(7)(2 \cdot 1 - 1) + 1 = 6 \cdot 7^0 \cdot \ln(7)(1) + 1 = 6 \ln(7) + 1$.

For the equation of the tangent line we put this together to get

$$y = 7 + (6 \ln(7) + 1)(x - 1).$$
CHAPTER 6. EXPONENTS AND LOGARITHMS BASE B

Problem 43 Compute the derivatives of the following.

a. \( f(x) = 3^{5x} \)

b. \( f(x) = 7 \cdot 9^{5x} \)

c. \( f(x) = 5 \cdot 7^{5x} + 3x \)

d. \( f(x) = 3^{5x+7} \)

e. \( f(x) = 11^{7x+5} \)

f. \( f(x) = 9^{x^2+5} \)

g. \( f(x) = 6^{2x^3} \)

h. \( f(x) = 7^{3x+x^2} \)

i. \( h(x) = 8^{\sin x} \)

j. \( h(x) = 7^{\cos x} \)

k. \( f(x) = 5^{\sin(x+1)} \)

l. \( f(x) = 4^{\tan x} \)

m. \( f(x) = 3^x \sin(x) \)

n. \( f(x) = \frac{7^x - 7^{-x}}{2} \)

o. \( f(x) = \frac{5^x + 5^{-x}}{2} \)

p. \( f(x) = \frac{3^x - 3^{-x}}{3^x + 3^{-x}} \)

Problem 44

Find the equation of the line tangent to the graph of \( f(x) = 7^{5x} \) above the value \( a = 0 \)

Problem 45

Find the equation of the line tangent to the graph of \( f(x) = 5^{x^2+5} \) above the value \( a = 0 \)

Problem 46

Find the equation of the line tangent to the graph of \( f(x) = 3^{x^2-5} \) above the value \( a = 1 \)

Problem 47

Find the equation of the line tangent to the graph of \( f(x) = 7^{x^2-5} \) above the value \( a = 2 \)
6.2 Solving Exponential and Logarithm Problems

In this section we exploit the fact that \( f(x) = e^x \) and \( g(x) = \ln(x) \) are inverses to get solutions to some problems.

Example 48 Solve \( e^{2x} = 5 \) for \( x \).

**Answer:** Since \( \ln(e^x) = x \) we take \( \ln \) of both sides to get.

\[
\ln(e^{2x}) = \ln(5).
\]

Because \( \ln(e^{2x}) = 2x \) this is saying.

\[
2x = \ln(e^{2x}) = \ln(5)
\]

or just

\[
x = \frac{1}{2} \ln(5).
\]

And that is our answer. \( \square \)

Example 49 Solve \( 7e^{4t} = 33 \) for \( t \).

**Answer:** As before we will take \( \ln \) of both sides. But first, we need to get the \( e^{stuff} \) by itself, so we divide by 7 to get.

\[
e^{4t} = \frac{33}{7}.
\]

Because \( \ln(e^{4t}) = 4t \) this is saying.

\[
4t = \ln(e^{4t}) = \ln(33/7)
\]

or just

\[
4t = \ln(33/7).
\]
Dividing by 4 we get
\[ x = \frac{1}{4} \ln(33/7). \]

And that is our answer.

Now we want to construct an exponential equation that is the analogue of a line,
\[ f(t) = Ke^{mt} \]

**Example 50** Find the values of \( K \) and \( m \) such that \( f(0) = 2 \) and \( f(3) = 10 \).

**Answer:** First, because \( f(0) = 2 \), it is easy to find \( K \).

\[
2 = f(0) = Ke^{m \cdot 0} = Ke^0 = K \cdot 1 = K.
\]

Next, we find \( m \) Beginning with \( 10 = f(3) = 2e^{m \cdot 3} \), we divide by 2 to get
\[
\frac{10}{2} = e^{m \cdot 3} \quad \text{or} \quad 5 = e^{m \cdot 3}.
\]

Now solving for \( m \) we take the ln of both sides to get
\[
\ln(5) = \ln(e^{m \cdot 3}) = m \cdot 3
\]

Dividing by 3 we get
\[
\frac{\ln(5)}{3} = m
\]

We have found the values for \( K \) and \( m \); we are done.

This problem was relatively easy because one of the numbers was zero. Let us try something more difficult.

**Example 51** Find the values of \( K \) and \( m \) such that \( f(1) = 2 \) and \( f(3) = 10 \).

**Answer:** We begin by writing down the two equations with the indicated values plugged in.

\[
2 = f(1) = Ke^{m \cdot 1} = Ke^m \quad \text{and} \\
10 = f(3) = Ke^{m \cdot 3} = Ke^{3m}.
\]
6.2. SOLVING EXPONENTIAL AND LOGARITHM PROBLEMS

Divide one by the other (we divide the top by the bottom).

\[ \frac{10}{2} = \frac{Ke^{3m}}{Ke^{m}}. \]

Now we use the property of exponents, \( \frac{e^x}{e^y} = e^{x-y} \), to write

\[ 5 = \frac{10}{2} = \frac{Ke^{3m}}{Ke^{m}} = e^{3m-m} \]

and factor out the \( m \)

\[ 5 = e^{3m-m} = e^{m(3-1)} = e^{2m}. \]

This we know how to solve; take the natural logarithm

\[ \ln(5) = \ln(e^{2m}) = 2m. \]

and divide by 2.

\[ m = \frac{1}{2} \ln(5). \]

Now that we have \( m \) we can go back and use it in something like the first equation,

\[ 2 = f(1) = Ke^{m-1} = Ke^{m} \]

to solve for \( K \). Notice what happens if we square both sides of the above equation,

\[ 2^2 = (Ke^m)^2 = K^2 \cdot e^{2m}. \]

Also notice that we already solved for \( 2m \), we got \( 2m = \ln(5) \). When we put the \( \ln(5) \) into the exponent of \( e \) we get

\[ 2^2 = K^2 \cdot e^{2m} = K^2 e^{\ln(5)} = K^2 5. \]

Or

\[ 4 = K^2 \cdot 5. \]

Dividing both sides by 4, we get

\[ \frac{4}{5} = K^2. \]
or $K = \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}}$.

So, there are our answers:

$$m = \frac{1}{2} \ln(5) \quad \text{and} \quad K = \frac{2}{\sqrt{5}}.$$ 

How do we write the function $f(t) = Ke^{mt}$?

One way to do this is use the $\exp(x)$ function. What is this function? It is just another way to write $e^x$. For example, $\exp(3) = e^3$ and another example $\exp(\ln(x)) = x$. In short $\exp(x) = e^x$. In other words, we have two different names for the same function. What’s the point in that? Well, you may have noticed that $m$ is a little complicated,

$$m = \frac{1}{2} \ln(5).$$

If we try to plug $m$ into $e^{mt}$ we get $e^\frac{1}{2} \ln(5)t$. It is a little hard to read. It is much easier to read

$$\exp(\frac{1}{2} \ln(5)t).$$

Using the exp notation our original function looks like

$$f(t) = Ke^{mt}$$

and with our solutions for $K$ and $m$ we get

$$f(t) = \frac{2}{\sqrt{5}} \exp(\frac{1}{2} \ln(5)t) = \frac{2}{\sqrt{5}} \exp(t \ln(\sqrt{5})).$$

The graph of $f(x)$ is shown in Figure 6.1

**Problem 52**

Solve the following for $x$. \hspace{1cm} $6 \cdot e^{2x} = 21$

**Problem 53**

Solve the following for $x$. \hspace{1cm} $3 \cdot e^{2x} = 21$

**Problem 54**

Solve the following for $x$. \hspace{1cm} $11 \cdot e^{3x} + 2 = 13$
6.2. SOLVING EXPONENTIAL AND LOGARITHM PROBLEMS

Figure 6.1: The Graph of $f(x) = \frac{2}{\sqrt{5}} \exp(t \ln(\sqrt{5}))$.

Problem 55
Solve the following for $x$. $5 \cdot e^{2x} - 4 = 6$

Problem 56
Solve the following for $x$. $5 \cdot e^{0.2x} = 13 \cdot e^{0.3x}$

Problem 57
In the function $f(t) = Ke^{mt}$ find the values of $K$ and $m$ such that $f(0) = 1$ and $f(3) = 7$.

Problem 58
In the function $f(t) = Ke^{mt}$ find the values of $K$ and $m$ such that $f(0) = 2$ and $f(5) = 9$.

Problem 59
In the function $f(t) = Ke^{mt}$ find the values of $K$ and $m$ such that $f(2) = 1$ and $f(5) = 11$.

Problem 60
In the function $f(t) = Ke^{mt}$ find the values of $K$ and $m$ such that $f(1) = 3$ and $f(3) = 7$. 
6.3 Inverses of base $b$ exponential functions

We have the natural logarithm, $\ln(x)$, but this is not the only logarithm. Recall that if $b > 1$, then the graph of $f(x) = b^x$ is an increasing function.

Figure 6.2: The Graph of $f(x) = b^x$.

Because $f_b(x) = b^x$ is increasing (recall $b > 1$) it has an inverse $g(x)$. That is

$$x = f_b(g(x)) = b^{g(x)} \quad \text{and} \quad x = g(f_b(x)) = g(b^x)$$

Notice different $b$ produce different functions. That is, $f(x) = 2^x$ if different from $f(x) = 3^x$. To distinguish we will sometimes write $f_b(x) = b^x$. So, $f_2(x) = 2^x$. Likewise, each inverse is different, so the inverse of $f_b(x) = b^x$ is frequently written $g_b(x)$.

In any case, by our previous discussions of inverses, we know several things about the function $g_b(x)$.

We know what the graph of $g_b(x)$ looks like (the reflection across the line $y = x$ of $b^x$).

We know its derivative! We do? We do. Its derivative is

$$g'_b(x) = \frac{1}{x \ln(b)}$$

and the generalized logarithmic derivative is, for $h(x) = g_b(Q(x))$,

$$h'(x) = \frac{Q'(x)}{Q(x) \cdot \ln(b)} = \frac{Q'(x)}{\ln(b) \cdot Q(x)}.$$

Why is that the derivative? Hang on a second; we will answer that shortly. First we would like to do a couple examples.
6.4. THE DERIVATIVE OF THE INVERSE OF THE BASE B EXPONENTIAL FUNCTION

Example 61  Let \( h(x) = g_4(x) \). (That is, \( 4^{g_4(x)} = x \)) Find \( h'(x) \).

Solution: We follow the rule directly.

\[
h'(x) = g'_4(x) = \frac{1}{x \ln(4)}.
\]

Done \( \square \)

Example 62  Let \( h(x) = g_4(\sqrt{x^2 + 1}) \) (where, \( 4^{g_4(x)} = x \)) Find \( h'(x) \).

Solution: We follow the (second) rule directly.

\[
h'(x) = g'_4(\sqrt{x^2 + 1}) = \frac{2x}{\ln(4) \cdot (x^2 + 1)}.
\]

Done \( \square \)

Problem 63  Compute the derivatives of the following functions.

\[
\begin{align*}
a. \ f(x) &= g_2(x) \\
b. \ f(x) &= g_3(x) \\
c. \ f(x) &= g_5(x) \\
d. \ f(x) &= g_8(x) \\
e. \ f(x) &= g_e(x) \\
f. \ f(x) &= g_9(x) \\
g. \ f(x) &= g_{10}(x) \\
h. \ f(x) &= g_3(x^2 - x) \\
i. \ f(x) &= g_3(\sqrt{1 + x^2}) \\
j. \ f(x) &= g_3(x + e^{2x}) \\
k. \ f(x) &= g_3(\frac{x}{2}) \\
l. \ f(x) &= g_3(\ln(x)) \\
m. \ f(x) &= g_3(x^2 \ln(x)) \\
n. \ f(x) &= g_3(g_2(x))
\end{align*}
\]

6.4  The derivative of the inverse of the base b exponential function

Now, let’s discuss why the derivative of \( g_b(x) \) is what it is. Remember we know the derivative of \( f_b(x) = b^x \).

\[
f'_b(\Box) = b^{\Box} \cdot \ln(b).
\]
We also know the chain rule, if \( h(x) = f(g(x)) \), then \( h'(x) = f'(g(x)) \cdot g'(x) \). We have the special case of \( h(x) = b^{g(x)} = x \). We apply the chain rule

\[
h'(x) = f'(g_b(x)) \cdot g'_b(x) = b^{g_b(x)} \ln(b) \cdot g'_b(x) = 1
\]

the 1 coming from the derivative of \( x \). So we have

\[
b^{g_b(x)} \ln(b) \cdot g'_b(x) = 1
\]

and recalling that \( b^{g_b(x)} = x \) we substitute to get

\[
x \ln(b) \cdot g'_b(x) = 1.
\]

We divide both sides of this by \( x \ln(b) \) to get

\[
g'_b(x) = \frac{1}{x \ln(b)}.
\]

And there we have it, a formula for the derivative of \( g_b(x) \).

### 6.5 The base b logarithm and its derivative

OK, OK, as some of you know this function \( g_b(x) \) has a more traditional name. Right? Traditionally

\[
g_b(x)
\]

is called

\[
\log_b(x).
\]

That’s right,

\[
\log_b(x) \equiv g_b(x).
\]

What that means is that what we learned about the derivative of \( g_b(x) \) also holds for \( \log_b(x) \).

\[
\text{If } h(x) = \log_b(x), \text{ then } h'(x) = \frac{1}{\ln(b) \cdot x}.
\]
And

\[
\text{If } h(x) = \log_b(f(x)), \text{ then } h'(x) = \frac{f'(x)}{\ln(b) \cdot f(x)}.
\]

We do a couple examples.

**Example 64** Let \( h(x) = \log_4(x) \). Find \( h'(x) \).

**Solution:** We follow the rule directly.

\[
h'(x) = \log'_4(x) = \frac{1}{x \ln(4)}.
\]

Done \( \square \)

**Example 65** Let \( h(x) = \log_4(\sqrt{x^2 + 1}) \). Find \( h'(x) \).

**Solution:** We follow the (second) rule directly.

\[
h'(x) = \log'_4(\sqrt{x^2 + 1}) = \frac{2x}{\ln(4) \cdot (x^2 + 1)}.
\]

Done \( \square \)

**Problem 66**

Compute the derivatives of the following functions.

a. \( f(x) = \log_2(x) \)  
   h. \( f(x) = \log_3(x^2 - x) \)

b. \( f(x) = \log_3(x) \)  
   i. \( f(x) = \log_4(\sqrt{1 + x^2}) \)

c. \( f(x) = \log_5(x) \)  
   j. \( f(x) = \log_7(x + e^{2x}) \)

d. \( f(x) = \log_8(x) \)  
   k. \( f(x) = \log_3(\frac{1}{x}) \)

e. \( f(x) = \log_e(x) \)  
   l. \( f(x) = \log_2(\ln(x)) \)

f. \( f(x) = \log_{10}(x) \)  
   m. \( f(x) = \log_7(x^2 \ln(x)) \)

g. \( f(x) = \log_{10}(x) \)  
   n. \( f(x) = \log_2(\log_2(x)) \)
6.6 Back to Tangent Lines

Example 67 Find the equation of the line tangent to the function \( h(x) = \log_2(x^2 + 1) \) above the value \( a = 1 \).

Solution. We need a point and a slope. The point is \((1, h(1))\). What do we use for \( h(1) \)?

Well, plugging 1 in we get \( h(1) = \log_2(1^2 + 1) = \log_2(2) = \log_2(2^1) = 1 \).

The slope is \( m = h'(1) \). From the generalized logarithmic differentiation formula we have

\[
h'(x) = \frac{2x}{\ln(2)(x^2 + 1)}.
\]

So we have \( h'(1) = \frac{2 \cdot 1}{\ln(2)(1^2 + 1)} = \frac{2}{\ln(2)2} = \frac{1}{\ln(2)} \).

Together, we have \( y - 1 = \frac{1}{\ln(2)} \cdot (x - 1) \).

Problem 68

Find the equation of the line tangent to the graph of \( f(x) = \log_2(x) \) above the value \( a = 2 \)

Problem 69

Find the equation of the line tangent to the graph of \( f(x) = \log_3(x + 1) \) above the value \( a = 8 \)
6.7 Computing the base b logarithm

\[ \log_b(x) = a \quad \text{if and only if} \quad b^a = x \]

Example 70  What is \( \log_2(4) \)?

**Answer:** Let \( a = \log_2(4) \). Then \( 2^a = 4 \) and since \( 4 = 2^2 \), it follows that \( 2^a = 2^2 \) and hence that \( a = 2 \).

Example 71  What is \( \log_5\left(\frac{1}{125}\right) \)?

**Answer:** Let \( a = \log_5\left(\frac{1}{125}\right) \). Then \( 5^a = \frac{1}{125} \) and since \( 125 = 5^3 \), then \( \frac{1}{125} = 5^{-3} \) and it follows that \( 5^a = 5^{-3} \) and hence that \( a = -3 \).

Example 72  What is \( \log_{1/5}\left(\frac{1}{125}\right) \)?

**Answer:** Let \( a = \log_{1/5}\left(\frac{1}{125}\right) \). Then \( \left(\frac{1}{5}\right)^a = \frac{1}{125} \) and since \( 125 = 5^3 \), then \( \frac{1}{125} = \left(\frac{1}{5}\right)^3 \) and it follows that \( \left(\frac{1}{5}\right)^a = \left(\frac{1}{5}\right)^3 \) and hence that \( a = 3 \).

**Problems:** What are the exact values of the following?

1. \( \log_2(8) = \)
2. \( \log_2(\sqrt{2}) = \)
3. \( \log_3\left(\frac{1}{\sqrt{3}}\right) = \)
4. \( \log_5(625) = \)
5. \( \log_7\left(\sqrt[7]{7}\right) = \)
6. \( \log_e(\sqrt[2]{4}) = \)
7. \( \ln(e^3) = \)

6.8 Properties of the base b logarithm

a. \( \log_b(1) = 0 \)

b. \( \log_b(b) = 1 \)

c. \( \log_b(x \cdot y) = \log(x) + \log(y) \)

d. \( \log_b(x/y) = \log(x) - \log(y) \)

e. \( \log_b(x^r) = r \log_b(x) \)

f. \( \log_b(a) = \frac{\log_e(a)}{\log_e(b)} \)

These properties follow from analogous exponential properties, namely.
g. \( b^0 = 1 \)

h. \( b^1 = b \)

i. \( b^m \cdot b^n = b^{m+n} \)

j. \( b^m/b^n = b^{m-n} \)

k. \( (b^n)^r = b^{rn} \)

l. \( b^{\log_b(a)} = a \)

Let us show c. is true.

**Proof.** Let \( x = b^m \) and \( y = b^n \). Then

\[
\log_b(x \cdot y) = \log_b(b^m \cdot b^n) = \log_b(b^{m+n}) = m + n.
\]

And since \( x = b^m \) we have that \( \log_b(x) = \log_b(b^m) = m \) and likewise, \( n = \log_b(b^n) = \log_b(y) \).

So

\[
\log_b(x \cdot y) = n + m = \log_b(x) + \log_b(y).
\]

We show f. is true.

**Proof.** Beginning with l. we take \( \log_c \) of both sides to get \( \log_c(b^{\log_b(a)}) = \log_c(a) \). Next, we use k. to write

\[
\log_b(a) \cdot \log_c(b) = \log_c(b^{\log_b(a)}) = \log_c(a).
\]

Dividing by \( \log_b(a) \) gives us f.

\[
\log_c(a) = \frac{\log_c(a)}{\log_c(b)}.
\]

**Example 73** Use the properties of logarithms to write \( \log(x) + \log(2) + \log(x+1) - \log(x^2-1) \) using just one logarithm.

**Answer:** Using property c. two times we can write

\[
\log(x) + \log(2) + \log(x+1) - \log(x^2-1) = \log(2x(x+1)) - \log(x^2-1).
\]

Then we use property d. to write

\[
\log(2x(x+1)) - \log(x^2-1) = \log\left(\frac{2x(x+1)}{x^2-1}\right).
\]

Then we factor \( x^2 - 1 = (x+1)(x-1) \) and rewrite \( \frac{2x(x+1)}{x^2-1} = \frac{2x(x+1)}{(x-1)(x+1)} = \frac{2x}{x-1} \). Putting it all together we get

\[
\log(x) + \log(2) + \log(x+1) - \log(x^2-1) = \log\left(\frac{2x}{x-1}\right).
\]
6.9. **SOLVING LOGARITHMIC EQUATIONS**

**Example 74**  Demonstrate how to find the exact value of $\log_7(98/3) + \log_7(3) - \log_7(2)$.

**Answer:** We use the properties of logarithms to combine the logarithms.

$$\log_7(98/3) + \log_7(3) - \log_7(2) = \log_7\left(\frac{98}{3} \cdot \frac{3}{2}\right).$$

Then we simplify the number $\frac{98}{3} \cdot \frac{3}{2} = 49 = 7^2$ giving us

$$\log_7(98/3) + \log_7(3) - \log_7(2) = \log_7(7^2) = 2.$$

**Problems:**

a. Write $\log(x^2(x - 1)) - \log(x) + \log(5)$ using a single log

b. Demonstrate how to find the exact value of $\log_5(325) - \log_5(13)$

---

6.9 **Solving logarithmic equations**

**Problems:**

a. Solve the following for $x$. $\log_3(x^2 - 3x - 7) = 1$

b. Solve the following for $x$. $\log_{10}(x^2 - 4x + 14) = 1$

c. Solve the following for $x$: $\log(2x + 1) = 2$

d. Solve the following for $x$: $\log(x - 1) = 2$

---

6.10 **Optimization with Exponents and Logarithms**

**Example 75**  Find the minimum of $f(x) = x \ln(x)$.

**Answer:** To find the minimum we take the derivative $f'(x)$ and set it equal to zero. We get an answer $x = a$. Then we plug $a$ into the second derivative $f''(a)$. If $f''(a) > 0$, then $a$ is a minimum and $f(a)$ is a locally minimum value for $f(x)$.

We make it so. $f'(x) = \ln(x) + x \cdot \frac{1}{x} = \ln(x) + 1$. Setting $f'(x) = 0$ we get

$$\ln(x) + 1 = 0,$$
or \( \ln(x) = -1 \). Raising \( e \) to both sides we get

\[
e^{\ln(x)} = e^{-1},
\]

or \( x = e^{-1} = \frac{1}{e} \).

Next we determine \( f''(x) = \frac{1}{x} \). Plugging \( a = \frac{1}{e} \) in we get

\[
f''(\frac{1}{e}) = \frac{1}{\frac{1}{e}} = e > 0.
\]

So \( f''(a) > 0 \) and thus \( a = \frac{1}{e} \) is where the minimum is and \( f(\frac{1}{e}) = \frac{1}{e} \ln(\frac{1}{e}) = -\frac{1}{e} \) is the minimum value of \( f(x) \).

**Problems:** Find the local extrema of the following. Use the 2nd derivative test to determine whether the extrema is a minimum or a maximum.

1. \( f(x) = (x - 1) \ln(x - 1) \)
2. \( f(x) = (x - 1) \log_2(x - 1) \)
3. \( f(x) = \log(x - 2) \log_5(x) \)
4. \( f(x) = \frac{e^x}{x} \)
5. \( f(x) = \frac{2^x}{3x} \)
6. \( f(x) = \frac{2^x}{x - 1} \)
6.11  Applications of logarithms and exponents

6.11.1  Population models

\[ P(t) = Ae^{kt} \] with \( P(t = 0) \) the initial population. By plugging \( t = 0 \) we can see that
\[ Ae^{k \cdot 0} = Ae^0 = A \] is the initial population. \( K \) is the relative periodic growth rate.

\[ P(t) = Ae^{kt} \] with the population known at two times \( P(t_1) = P_1 \) and \( P(t_2) = P_2 \).

\[ P(t) = Ab^t \] with \( P(t = 0) \) the initial population.

\[ P(t) = Ab^t \] with the population known at two times \( P(t_1) = P_1 \) and \( P(t_2) = P_2 \).

The average growth rate of a population between times \( t_1 \) and \( t_2 \) is
\[
\frac{P(t_2) - P(t_1)}{t_2 - t_1}.
\]

The instantaneous population growth rate at time \( T \) is \( f'(T) \).

6.11.2  Data analysis and Population models

If \( P(t) = Ae^{Kt} \) and we have population/time data in the form of pairs of numbers \((t_o, P_o)\) we may want to find values of \( A \) and \( K \) so that \( P(t) = Ae^{Kt} \) is a good representation of the data.

Notice
\[
\ln(P(t)) = \ln(Ae^{Kt}) = \ln(A) + \ln(e^{Kt}) = \ln(A) + Kt.
\]

If we let \( Q = \ln(P) \), then that equation looks like the following.

\[ Q(t) = \alpha + Kt \]

where \( \alpha = \ln(A) \).

If we take the population data \((t_o, P_o)\) and change the data into \((t_o, \ln(P_o))\) by taking the natural logarithm of the population numbers, then we expect the data to lie on the line \( Q = \alpha + Kt \). So, we plot the \((t_o, \ln(P_o))\) data points and find a best fit line. From the best fit line we read off the slope, \( K \), and the intercept \( \alpha \). We solve for \( A \) using \( \alpha = \ln(A) \). Careful, \( K \) is not how much the population ratio will change in one year.
Problems:

1. A population of bacteria is observed to have about 5200 cells in it at 7AM and at 5PM it is observed to have 7200 cells. Construct an exponential model of the bacteria growth. Use this model to predict how many bacteria cells there will be the next day at 7AM.

6.12 Area and the antiderivative of $1/x$ and beyond

We now calculate the values of expressions like the following.

$$\int_1^4 \frac{2x^2 + x}{x} \, dx$$

We know that if we can find the antiderivative $F(x)$ then the answer is just $F(4) - F(1)$. We also know the antiderivative of the $x^2$ is $x^3/3$ so the answer comes down to figuring out the antiderivative of $\frac{x}{(x^2 + 1)}$.

Recall we know that if $f(x) = \ln(x)$, then $f'(x) = \frac{1}{x}$. This tells us that if $g(x) = \frac{1}{x}$, then $G(x) = \ln(x) + C$. So

$$f(x) = \frac{1}{x} \quad \text{implies} \quad G(x) = \ln(x) + C.$$ 

Example 76 Find the antiderivative of $f(x) = \frac{x}{(x^2 + 1)}$.

Answer: We use substitution. Let $u = x^2 + 1$, then $u' = 2x$, so $\frac{1}{2}u' = x$. Then

$$\tilde{f}(u) = \frac{1}{2} \cdot \frac{u'}{u}.$$ 

Changing $u$ from a function to a variable makes $u' = 1$, giving us

$$\tilde{f}(u) = \frac{1}{2} \cdot \frac{1}{u} = \frac{1}{2u}.$$ 

The antiderivative of this is

$$\tilde{F}(u) = \frac{1}{2} \ln(u) + C.$$
Back-substituting gives us
\[ F(x) = \frac{1}{2} \ln(x^2 + 1) + C. \]
Done!

Thus we can answer our original question
\[
\int_1^4 2x^2 + \frac{x}{(x^2 + 1)} \, dx = \left( \frac{2x^3}{3} + \frac{1}{2} \ln(x^2 + 1) \right)_1^4 \\
= \frac{2 \cdot 4^3}{3} + \frac{1}{2} \ln(4^2 + 1) - \left( \frac{2 \cdot 1^3}{3} + \frac{1}{2} \ln(1^2 + 1) \right) \\
= \frac{2 \cdot 63}{3} + \frac{\ln(17/2)}{2}
\]

Problems:

Find the following antiderivatives. Use substitution if necessary.

1. \( f(x) = \frac{7}{x} + x^2 \)
2. \( \int \frac{6x^2 + 2}{x^3 + x} \, dx = \)
3. \( \int \frac{3e^{3x}}{e^{3x} + 1} \, dx = \)
4. \( \int \frac{e^x + 2}{e^x + 2x} \, dx = \)
5. \( \int \frac{x + 1}{x^2 + 2x - 1} \, dx = \)
6. \( \int \frac{1}{x \ln x} \, dx = \)

Find the following areas.

7. \( \int_{-1}^1 \frac{3e^{3x}}{e^{3x} + 1} \, dx = \)
8. \( \int_0^2 \frac{x + 1}{x^2 + 2x - 1} \, dx = \)

### 6.13 A natural logarithm derivative tidbit

Recall \( a = \ln(x) \) doesn’t make sense if \( x \) is negative. (Why? Well, \( e > 0 \), so that would mean \( e^a = x \) is negative. So, some power of a positive number is negative. In short, it won’t happen.) However, if \( x \) is negative then \(-x\) is positive and thus \( g(x) = \ln(-x) \) makes sense.

Using the chain rule we have
\[ g'(x) = \frac{1}{-x} \cdot (-1) = \frac{1}{x}. \]
Notice there is no negative sign in the answer’s denominator, \( g'(x) = \frac{1}{x} \).

Consider \( h(x) = \ln |x| \), notice the absolute value signs. Recall

\[
|x| = \begin{cases} 
  x & \text{if } x \geq 0 \\
  -x & \text{if } x < 0 
\end{cases}
\]

So

\[
h(x) = \ln |x| = \begin{cases} 
  \ln(x) & \text{if } x > 0 \\
  \ln(-x) & \text{if } x < 0 
\end{cases}
\]

And so,

\[
h'(x) = \begin{cases} 
  \frac{1}{x} & \text{if } x > 0 \\
  \frac{1}{-x} & \text{if } x < 0 
\end{cases}
\]

It doesn’t matter whether \( x > 0 \) or whether \( x < 0 \) the derivative \( h'(x) = \frac{1}{x} \) has the same form! In summary.

\[
\text{If } h'|x| = \ln(x), \text{ then } h'(x) = \frac{1}{x}.
\]

### 6.14 The graphs of the logarithm

Recall the graph of \( b^x \) for \( b > 1 \) (see Figure 4.2). Because the graph of \( \log_b(x) \) is the graph of \( b^x \) reflected across the line \( y = x \), it is clear that the graph looks like that shown in Figure 6.4.

**Problems:**
Figure 6.4: The Graph of \( f(x) = \log_b(x) \) for \( b > 1 \).

1. Sketch \( \log_3(x) \) labelling at least 2 points and any asymptotes.

2. Sketch \( \log_5(x) \) labelling at least 2 points and any asymptotes.

3. Sketch \( \log_\pi(x) \) labelling at least 2 points and any asymptotes.
4. Sketch $\log_{1/4}(x)$ labelling at least 2 points and any asymptotes.